Math 696

Friday, February 28, 2025 8:58 AM

Generalized cohomologies (spectra) related to number theory

? Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$ + properties

<u>Class field theory</u>: Gal$(\bar{\mathbb{Q}}/\mathbb{Q})^{Ab}$ ← Galois extensions of $\mathbb{Q}$ with abelian Galois group + divisibility properties

<u>Tate & others</u>: $\underbrace{\text{local class field thy.}}_{}$

connected with stable homotopy thy via FGL

p-adic integers: $\mathbb{Z}_p = \varprojlim (\mathbb{Z}/p^n)$

field of fractions: $\mathbb{Q}_p$

the question is easier for $\mathbb{Q}_p$ (Gal$(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is now known)

finite

$$\begin{array}{ccc} \mathbb{Q} & \subset & K \\ \cup & & \cup \\ \mathbb{Z} & \subset & \mathcal{O}_K \end{array}$$

K ← number field

$\subset$ ↑ integral

$\mathcal{O}_K$ ← divisibility

$\mathbb{Z}$ ↑ prime p

↑ product of powers of primes

often we don't have uniqueness of divisibility: prime – prime ideal local class field theory : $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)^{Ab}$

local field : finite extension K of $\mathbb{Q}_p$

(of char. 0)

$\boxed{Gal(\bar{K}/K) \text{ "=" } K^\times}$

discrete group

$(K \setminus \{0\}, \cdot)$

local reciprocity

pro finite group

(inverse limit of finite groups - compact)

$$\begin{array}{ccc} \mathbb{Q}_p & \subset & K \\ \cup & & \cup \\ \mathbb{Z}_p & \subset & \mathcal{O}_K \end{array}$$

max. integral

? divisibility properties of $\mathcal{O}_K$.

only has one prime $\sqrt[n]{p}$ $n=1$: K unramified.

Unramified extensions of $\mathbb{Q}_p$:

unramified

$$\begin{array}{ccc} \mathbb{Q}_p & \subset & K \\ \cup & & \cup \\ \mathbb{Z}_p & \longrightarrow & \mathcal{O}_K \\ \downarrow & & \downarrow \\ \mathbb{F}_p & \subset & \mathbb{F}_{p^k} \end{array}$$

$Gal = \mathbb{Z}/k$

lift the minimal polynomial

$Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p) = \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$

Totally ramified extensions are more mysterious.

Assume K is unramified over $\mathbb{Q}_p$.

"explains" v_n, all content in homotopy theory

$K \subset L$

for elements, Eisenstein series:

$p(x)$

power series with coefficient in $\mathcal{O}_K$

$\sqrt[\ell]{\pi}$ } ideals

prime

degree of the extension

$L = K[x]/p(x)$.

$p(x) \equiv p \quad (x^2)$

$\equiv x^\ell \quad (p)$.

Which of these totally ramified extensions are Galois?

Which have an abelian Galois group?

Related to FGL - <u>Lubin-Tate theory</u>

Let K be an unramified extension of $\mathbb{Q}_p$ of degree n.

Let $[p](x)$ ← "Secret": This will be a p-series of an FGL

be a power series in x:

$$[p](x) \equiv px \mod (x^2) \qquad (*)$$
$$\equiv x^{p^n} \mod (p).$$

Lemma (Lubin-Tate): Choosing $a_1, \dots a_m \in \mathcal{O}_K$, there exist a unique series $f(x_1, \dots x_m)$ such that

$$f(x_1, \dots x_m) \equiv a_1 x_1 + \dots + a_m x_m \ (x_1, \dots x_m)^2$$
$$f([p]x_1, \dots [p]x_m) = [p] f(x_1, \dots x_m).$$

If we just took $f(x_1, \dots x_n) := a_1 x_1 + \dots + a_n x_n$, we are already OK mod p.

Fixing higher degree monomials of $x_1, \dots x_n$ will be OK because of $(*)$ and because we are OK mod p. □

$m = 2 \quad a_1 = a_2 = 1 \qquad f(x,y)$ is an FGL. ($\Leftarrow$ uniqueness)

$\qquad\qquad\qquad\qquad\qquad\qquad\ \ x +_F y$ ← Lubin-Tate law

For the same reason, the Lubin-Tate law is an $\mathcal{O}_K$-formal module.

$\alpha \in \mathcal{O}_K$ I have a series $[\alpha]X = \alpha X + \text{HOT}$

$$[\alpha](X +_F Y) = [\alpha]X +_F [\alpha]Y$$

apply the lemma for $m = 1$, $a_1 = \alpha$.

Note: $\alpha \in \mathbb{N}$

$$[\alpha]X = \underbrace{X +_F \cdots +_F X}_{\alpha \text{ times}}$$

I can now construct a field extension:

$\dfrac{[p]X}{X}$ is an Eisenstein series: $K[x] \Big/ \left(\dfrac{[p]X}{X}\right)$

More generally,

$\dfrac{[p^\ell]X}{[p^{\ell-1}]X}$ is an Eisenstein series $L = \dfrac{[p^\ell]X}{[p^{\ell-1}]X}$

divisible: $[p^\ell]X = [p]\left([p^{\ell-1}]X\right)$

$$[L:K] = p^{\ell m} - p^{(\ell-1)m}$$

these contain all Galois abelian totally ramified extension of K

$$L = K[x] \Big/ \frac{[p^\ell]x}{[p^{\ell-1}]x}$$

$$[L:K] = p^{\ell n} - p^{(\ell-1)n}$$

$$\mathrm{Gal}(L/K) = \left(\mathcal{O}_K / p^\ell \mathcal{O}_K\right)^\times$$ ← group of units = invertible elements

α

$$\alpha(x) = [\alpha]x$$

<u>Constructive</u>
local class field theory

$\mathrm{Gal}(\bar{K}/K)^{ab}$ " = " $K^\times$

$\mathcal{O}_K^\times$ ← totally ramified side

$\mathbb{Z}$ ← unramified

---

decompose $(\pi_* S)^\wedge_p$ into parts related to $K(n)$

Chromatic homotopy theory: From MU, construct complex-oriented generalized cohomologies with the Lubin-Tate FGL related to v_n. ← attached to x^{p^n}

Class field theory:

no construction known

(Shimura theory
$\Rightarrow$ special values)

$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})^{ab} \text{ "=" } I(K)/K^\times$

$\bar{K}/K$

$K \longrightarrow \mathbb{A}(K) = \left\{ x \in \prod_{\substack{p \text{ prime} \\ \text{in } \mathcal{O}_K}} \hat{K}_p \;\middle|\; \begin{array}{l} \text{all but finitely} \\ \text{many are in } \hat{\mathcal{O}}_{K,p} \end{array} \right\} \times \prod_{\mathbb{C}} \mathbb{R}$

infinite primes:

$K \subset \mathbb{R}$

$\mathbb{C}$

$\pi_0(\mathbb{R}^\times) = \mathbb{Z}/2$

connected component

$\downarrow$

$I(K) = \text{units in } \mathbb{A}(K)$