Math 696

Monday, March 10, 2025 9:02 AM

Lubin-Tate formal group law on $\mathcal{O}_K$ where K is a degree n unramified extension of $\mathbb{Q}_p$.

$$\begin{array}{ccccc} \mathbb{F}_p & \xleftarrow[\text{maximal}]{\text{mod } p} & \mathbb{Z}_p & \subset & \mathbb{Q}_p \\ \cap & \text{(integral} \rightarrow & \cap & & \cap \\ \mathbb{F}_{p^n} & \xleftarrow[\text{mod } p]{\text{extension}} & \mathcal{O}_K & \subset & K \end{array}$$

Lubin-Tate proved that given any series $\pi(x) \in \mathcal{O}_K[[x]]$

$$\pi(x) \equiv px \mod x^2$$
$$\pi(x) \equiv x^{p^n} \mod p$$

there exists a <u>unique</u> FGL on $\mathcal{O}_K$ such that $\pi(x) = [p]_F x := \underbrace{x +_F \cdots +_F x}_{p}$

This is also an $\mathcal{O}_K$-module FGL: For $\alpha \in \mathcal{O}_K$, there is a series $[\alpha] x = \alpha x + \text{HOT} \pmod{x^2}$ which satisfies formal distributivity:

$$[\alpha] x +_F [\beta] x = [\alpha+\beta]_F x$$
$$[\alpha](x +_F y) = [\alpha] x +_F [\alpha] y.$$

The only totally ramified Galois extensions of K with Abelian Galois group.

$\exists\, \pi \in \mathcal{O}_L \quad (\pi)^{[L:K]} = (p)$

Choose a natural number N. Then there is a totally ramified extension

$K \subset L$ $\quad$ $\deg = p^{Nn} - p^{n(N-1)}$ $\quad$ x

$$L = K[x] \Big/ \frac{[p^N] x}{[p^{N-1}] x}$$

Eisenstein series

$\equiv p \quad (x)$

$\equiv x^{p^{Nn}} \quad (p)$

$\mathrm{Gal}(L/K) \cong (\mathcal{O}_K / p^N \mathcal{O}_K)^\times$: $\quad \alpha(x) = [\alpha] x$

We can make a complex-oriented generalized cohomology with this FGL. Homotopical significance: Chromatic homotopy theory

height n related to v_n

How? For the Lubin-Tate law, it is convenient to consider it mod p: A FGL on $\mathbb{F}_{p^n}$, $[p]x = x^{p^n}$ (Honda FGL)

char. p, lowest term x^{p^n}
p-typical : height n

Lubin-Tate studied a universal deformation of the Honda FGL.

Category : Objects = local rings R which are p-complete, $R/p = \mathbb{F}_{p^n}$,
together with a formal group law Φ,
such that Φ mod p = Honda FGL.

Morphisms :
$$\begin{tikzcd} \overset{\Phi}{R_1} \arrow[rr, "h"] \arrow[dr] & & \overset{\psi}{R_2} \arrow[dl] \\ & \mathbb{F}_{p^n} & \end{tikzcd}$$

homomorphism of rings

together with a $\star$-isomorphisms: reparametrization of $h\Phi$ to ψ

FGL's on R_2

$a_0 x + a_1 x^2 + \dots$
$\equiv x$ mod p

__Theorem__ (Lubin-Tate): The above category has an initial object

$$R = \mathcal{O}_K[[u_1, \dots u_{n-1}]] \ , \quad \Phi$$

K = degree n unramified extension of $\mathbb{Q}_p$

related to $v_1, \dots v_{n-1}$ ← homotopy point of view

universal deformation

Recall: $|v_i| = 2(p^i - 1)$

What does completion mean?

We construct from MU (using spectral algebra) a spectrum E_n

$$\pi_* E_n = \mathcal{O}_K[[u_1, \dots u_{n-1}]][u, u^{-1}]$$

$|u_i| = 0$

$|u| = 2$

because of the FGL!

v_n " = " $u^{p^n - 1}$

$\underbrace{v_i \text{ " = " } u^{p^i - 1} u_i}_{\text{FGL theory identities}}$ $\quad i = 1, \dots, n-1$

Recall $v_i = x_{p^i - 1}$

$MU_* = \mathbb{Z}[x_1, x_2, \dots]$

Morava E-theory (in the 90's M.J. Hopkins, P. Goerss, E_n is an E_∞-ring spectrum)

$MU \xrightarrow{E_\infty} E_n$

commutative in a coherent way

Before E_n, there existed a "classical theory":

$\pi_* K(n) = \mathbb{Z}/p[v_n, v_n^{-1}]$

$E(n)$: $\pi_* E(n) = \mathbb{Z}_{(p)}[v_1, \dots v_n][v_n^{-1}]$. not known to be E_∞

From a homotopical point of view, $E(n)$ is "equivalent" to E_n. calculating $\pi_* S$

Bousfield localization

Back to the $\overbrace{\text{derived category}}^{\text{homotopy sense}}$ $D\mathrm{Spectra} = $ (Cell spectra, homotopy classes of morphisms)

weak equivalences become isomorphisms

What if I wanted more equivalences? Considering a spectrum E, an E-equivalence is a morphism in $D\mathrm{Spectra}$ $f: X \to Y$ inducing an $\cong$ in E-homology:

exactly isomorphisms if $E = S$.

$$E_* f : E_* X \xrightarrow{\cong} E_* Y$$

Is there a derived category $D_E\mathrm{Spectra}$ with respect to E-equivalences?

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Bousfield proved: $\mathcal{D}$(Spectra) has **localization** with respect to **E-equivalences**:

For every spectrum X, there exists an E-equivalence

$$X \xrightarrow{\sim_E} L_E X$$

(induces $\cong$ in E_*)

and $L_E X$ is **E-local**: If $f : Z \to Y$ is an E-equivalence ($E_* f : E_* X \xrightarrow{\cong} E_* Y$)

def. of an E-local object

then

$$\widetilde{[Y, L_E X]} \xrightarrow[{[f, L_E X]}]{\cong} [Z, L_E X] \leftarrow \mathrm{Hom}_{\mathcal{D}\text{Spectra}}(?,?) = [?,?]$$

Chromatic homotopy theory studies $\pi_* S$ via $\pi_* L_{E_n} S$

($L_{E_n} S_p^\wedge = L_{E(n)} S_p^\wedge$)

Spectral sequence: Chromatic spectral sequence

$$S_p^\wedge \xrightarrow{\sim} \mathrm{holim}\left(\cdots\cdots \to L_{E_2} S_p^\wedge \to L_{E_1} S_p^\wedge \to L_{E_0} S_p^\wedge\right)$$

chromatic convergence theorem

we can essentially almost reduce $\pi_* S$ to group cohomology (+ spectral sequence)