Math 696

Wednesday, March 12, 2025 9:01 AM

Chromatic homotopy theory

depends on a prime p

DSpectra$_{(p)}$

$X \rightsquigarrow X \wedge M\mathbb{Z}_{(p)}$

The basic ingredients: Morava $K(n)$-theory:

$$K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$$

$|v_n| = 2(p^n - 1)$

$= MU/(p, x_i \,|\, i \neq p^n - 1)[x_{p^n-1}]$

$MU_* = \mathbb{Z}[x_1, x_2, \ldots]$

$|x_k| = 2k$

$v_n = x_{p^n-1}$

MU is an E_∞-ring spectrum

exists as an E_∞-module

FGL

p-typical

$[p]x = x^{p^n}$ ← height n } Honda FGL

Lubin-Tate deformation theory. We consider the Honda law on $\mathbb{F}_{p^n}$.

$\exists$ initial object L_n in the category of local p-complete rings R with FGL Φ

that $R/m = \mathbb{F}_{p^n}$, $\Phi/m = $ Honda FGL F.

max. ideal

morphisms of local rings

K degree n unramified extension of $\mathbb{Q}_p$

$L_n = \mathcal{O}_K[[u_1, \ldots, u_{n-1}]]$

"u_i is related to v_i"

Morphisms

$$\begin{tikzcd} (R, m) \arrow[rr, "f"] \arrow[dr] & & (R', m') \arrow[dl] \\ & \mathbb{F}_{p^n} & \end{tikzcd}$$

Φ, Φ'

$f_* \Phi \xrightarrow{\cong} \Phi'$

$*$-isomorphism

reparametrization

$h(x) = a_1 x + a_2 x^2 + \ldots$

$a_1 \in (R')^\times \qquad a_n \in R'$

$h(x) \bmod m' = x$

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Morava E_n - theory: complex-oriented spectrum

Hopkins 1990's } $E_{n*} = \mathcal{O}_K [[u_1, \dots u_{n-1}]][u, u^{-1}]$ $|u| = -2$

Connection with MU: $v_i = u_i u^{1-p^i}$ $i = 1, \dots n-1$

$v_n = u^{1-p^n}$.

Not exactly a quotient of MU_* by a regular sequence, but it is close:

$E(n)_* = \mathbb{Z}[v_1, \dots v_n][v_n^{-1}]$ kill x_i, if $p^j - 1$, $j = 1, \dots n$

invert $x_{p^n - 1}$

Take a $\vee$ (coproduct) of the right number of (suspended) copies, complete.

$h(x) +_{F_n} h(y)$

$a_1 \in \mathbb{F}_{p^n}^\times$ $h(x)$ $a_1 x + a_2 x^2 + \dots$ $= h(x +_{F_n} y)$

$E_{n*} = L_n [u, u^{-1}]$, $\mathbb{S}_n$ = group of automorphisms of the $\underbrace{\text{Honda}}_{[p] = x^{p^n}}$ FGL F_n on $\mathbb{F}_{p^n}$.

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Morava stabiliser group (pro-finite group $\underbrace{\text{pro-finite group}}_{\text{known}}$)

By universality, $\mathbb{S}_n$ acts on L_n.

One considers also $\mathbb{G}_n = \overbrace{Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)}^{\mathbb{Z}/n} \ltimes \mathbb{S}_n$ also "acts" on L_n in a "twisted" way

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By obstruction theory, Hopkins - Goerss proved: $\mathbb{G}_n$ acts on the spectrum E_n, which is a E_∞ - ring spectrum. ← by E_∞ - maps

It is interesting to study the homotopy fixed point

$$E_n^{hG_n}$$

some technical issues because G_n is a pro-finite group

Why is it interesting?

Bousfield localization: Bousfield proved that for any spectrum E, the derived category DSpectra has localization with respect to equivalences $\mathcal{E}_E$

$f: X \to Y \in \mathcal{E}_E$ when $E_* f : \underbrace{E_*}_{\pi_*(E \wedge X)} X \xrightarrow{\cong} E_* Y$ is an isomorphism.

$$\exists \quad X \xrightarrow[\in \mathcal{E}_E]{\sim_E} L_E X, \; L_E X \text{ is } \mathcal{E}_E\text{-local.}$$

Theorem (Hopkins et al.): $L_{K(n)} S = E_n^{hG_n}$.

$H_c^*(G_n; E_{n*}) \Rightarrow \pi_* E_n^{hG_n}$.

(Adams-Novikov spectral sequence: descent spectral sequence based on MU.)

Ravenel: Nilpotence and periodicity in stable homotopy theory

Why are we interested in $L_{K(n)}S$? ← we are in DSpectra$_{(p)}$

Ravenel proved the Chromatic convergence theorem:

uses ANSS $\{$ $S = \operatorname{holim}\left(\cdots \to L_{E_3}S \to L_{E_2}S \to L_{E_1}S \right)$

what are these maps?

$L_{E_n}X = L_{K(n) \vee K(n-1) \vee \cdots \vee K(0)}X$ ∴ E_n-equivalence ⇒ E_{n-1}-equivalence

$\underbrace{E_n\text{-equivalence}}_{E} \equiv \underbrace{K(n) \vee K(n-1) \vee \cdots \vee K(0)}_{F}\text{-equivalence}$ } we call E, F Bousfield-equivalent

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So we certainly are interested in $L_{E_n}S$. ← more complicated, less grounded than $L_{K(n)}S$.

<u>Another idea of Hopkins:</u> The chromatic square: X = p-localized finite spectrum (e.g. $S_{(p)}$)

$$\begin{tikzcd}
L_{E_n} X \arrow[r] \arrow[d] \arrow[dr, phantom, "h \lrcorner"] & L_{K(n)} X \arrow[d] \\
L_{E_{n-1}} X \arrow[r, "\alpha"'] & L_{E_{n-1}} L_{K(n)} X \arrow[l, dashed, bend right, "s"']
\end{tikzcd}$$

$K(n) \vee \cdots \vee \underbrace{K(0)}_{H\mathbb{Q}}$

homotopy pullback (i.e. cofibers of rows, equivalently columns are the same)

proof w/ ANSS

<u>Chromatic splitting conjecture</u> $\exists$ s such that $\alpha s \simeq \mathrm{Id}$.

Assuming this conjecture: $L_{K(n)} S \to L_{E_{n-1}} L_{K(n)} X \xrightarrow{s} L_{E_{n-1}} S \to L_{K(n-1)} S$

Diagram chasing would imply: $S = \operatorname{holim}\left(\cdots \to \underbrace{L_{K(n)} S}_{E_n^{hG_n}} \to L_{K(n-1)} S\right)$

One more aspect of the chromatic story:
Telescope conjecture.

Let X be a finite spectrum, fix prime p.

Theorem (Hopkins, Devinatz, Smith): (1) $K(n)_* X = 0 \Rightarrow K(n-1)_* X = 0$.

(2) $\exists$ some minimal n such that $K(n)_* X \neq 0$. $\leftarrow$ X is called type v_n-spectrum

(3) If X is a type v_n-spectrum, then there exists a v_n-self-map:

$$f : \underset{N>0}{\Sigma^N X} \to X \text{ which induces } \cong \text{ in } K(n)_*$$

Further, Cf is type v_{n+1}.

Telescope conjecture: If X type v_n, then $v_n^{-1} X$ is Bousfield-equivalent to E_n.

True for $n=1$, recently disproved by Burklund, Hahn, Levy, Schlank for $n>1$.