Toward equivariant homotopy theory (with group actions).

The first motivating example was K-theory.

The "rest of the story" of K-theory relates to the index theorem.

Let M be a smooth (C^∞) compact (real) manifold. Let E, F be smooth complex vector bundles on M of equal dimension k.

can be defined using the sheaf of smooth sections.

Ov: $(U_i)_{i\in I}$ open sets on which $E|_{U_i} = U_i \times \mathbb{C}^k$. Passage from i to j: $h_{ij}: U_i \cap U_j \to GL_k(\mathbb{C})$ be a smooth ($=C^\infty$) map. Cocycle condition: on $U_i \cap U_j \cap U_k$, $h_{ik} = h_{jk} \cdot h_{ij}$.

The notion of a differential operator $d: C^\infty(E) \to C^\infty(F)$

smooth sections

On each U_i, $d_i : C^\infty(U_i) \otimes_{\mathbb{R}} \mathbb{C}^k \to C^\infty(U_i) \otimes_{\mathbb{R}} \mathbb{C}^k$, given by a $(k\times k)$-matrix of operators of the form

WLOG, $U_i \subseteq \mathbb{R}^n$ open

$$\sum f_{j_1,\dots,j_n}(x_1,\dots,x_n) \frac{\partial^{j_1+\dots+j_n}}{\partial x_1^{j_1}\cdots \partial x_n^{j_n}}$$

finite sum over tuples $(j_1,\dots,j_n)$

smooth functions valued in $\mathbb{C}$

coordinates $x_1,\dots,x_n$

which transforms correctly with respect to the reparametrizations h_{ij}:

quite complicated, but familiar.

The <u>order</u> of the operator is $r = \max(j_1 + \dots + j_n)$.

invariant under reparametrization

We can define the <u>symbol</u> of the operator as

orders of the partial derivative

$$\left(\sum_{j_1+\dots+j_n=r} f_{j_1,\dots,j_n}(x_1,\dots x_n)\, y_1^{j_1}\cdots y_n^{j_n} \right)_{k\times k\text{ - matrix}} \qquad (*)$$

order

The operator is called <u>elliptic</u> when the matrix $(*)$ is invertible whenever $(y_1,\dots y_n) \neq (0,\dots,0)$.

A more invariant description of the symbol: What are the variables $y_1,\dots y_n$ geometrically? We can say $y_i = dx_i$.

coordinates of the cotangent bundle $\tau_M^\vee = \mathrm{Hom}(\tau_M, \mathbb{R})$ (dual of the tangent bundle).

(Even though the transformation of the differential operator is complicated, the "top part" transforms as a tensor of this type.) Denote by TM^* the total space of the dual tangent bundle of M. TM^* is a <u>weakly complex manifold</u>, which means that its tangent bundle is complex. (essentially, the sum of two copies of the same real bundle).

compact CW-cx

We can define, for an open set $U \subseteq Y$, where Y is a space, an element of $K^0(Y,U) = \tilde{K}^0(C(U \subseteq Y))$ by specifying some complex vector bundles E, F on Y and a homomorphism $f : E \to F$, which is isomorphic on U.

The ellipticity condition defines thus an element of

$$K^0(T^*M, \underbrace{T^*M \setminus M}_{dx_i := y_i \text{ coordinates are non-zero}}) \xrightarrow{\text{reduce}} \tilde{K}^0\left((T^*M)^\circ\right)$$

0-section

1-point compactification

Other models: put a Euclidean metric on $\tau_M^\vee$,
$K^0(D\tau_M^\vee, S\tau_M^\vee)$

collapse this into a point

One can also embed $M \subset \mathbb{R}^N$. We can similarly push forward

$$\tilde{K}^0(T^*M^\circ) \longrightarrow \tilde{K}^0(\underbrace{T^*\mathbb{R}^{N,\circ}}_{\text{sphere of even dimension}}) \cong \mathbb{Z}$$

Bott periodicity

Recall at the beginning, we had the elliptic operator $d : C^\infty(E) \to C^\infty(F)$.
The integer that we produce in this way is called the topological index

$\mathrm{index}_t(d)$.

Another part of the story (involves some PDE theory):

$\dim \underbrace{\ker d}_{\text{solution of a system of elliptic PDE's on a compact manifold}}$, $\dim \operatorname{Coker} d < \infty$

$C^\infty(E)$ is not a nice space. Sobolev norm: inner product, complete. Then we have a Hilbert space. Fredholm operator, easy to handle, enough for the index theorem.

One defines $\operatorname{index}_a(d) := \dim \ker d - \dim \operatorname{Coker} d$

PDE: "weak solution"

one proves that the weak solution is smooth.

Theorem (Atiyah): $\operatorname{index}_t(d) = \operatorname{index}_a(d)$.

Atiyah's proof: characterize index_t by certain axioms, and show that index_a satisfies them.

Next time: Maybe index_t in more detail (what is really happening in K-theory)

$\rightsquigarrow$ equivariant Bott periodicity $\rightsquigarrow$ equivariant generalized cohomology.