Math 696

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Index theorem via K-theory

M compact C^∞-real n-manifold E, F smooth complex bundles on M of equal dimension k. $d : \Gamma^\infty E \to \Gamma^\infty F$ is a differential operator. In local coordinates $x_1, \dots x_n$:

smooth sections

$$\sum_{i_1, \dots, i_n} f_{i_1, \dots i_n}(x_1, \dots x_n) \frac{\partial^{i_1 + \dots + i_n}}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} \qquad (*)$$

Order $r = \max(i_1 + \dots + i_n)$. Symbol: look at $i_1 + \dots + i_n = r$. Replace in $(*)$ $\dfrac{\partial^r}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}$ by $y_1^{i_1} \dots y_n^{i_n}$ ← new variables.

This transforms as a tensor, so we can write $y_i = dx_i$.

We consider the manifold T^*M = the total space of the dual of the tangent bundle of M. T^*M is a $\underbrace{\text{weakly complex manifold}}$.

The tangent bundle τ_{T^*M} comes with a cx. structure

"dx_j is an infinitesimal form of x_j"

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$M \subset T^*M \xrightarrow{\pi} M$

0-section

The operator d is called elliptic if the symbol is an invertible $k\times k$ complex matrix for $(y_1,\dots,y_n) \neq (0,\dots,0)$.

We pull back the complex k-bundles E, F to T^*M: π^*E, π^*F
The symbol gives a fiberwise $\mathbb{C}$-linear morphism

$$\varphi: \pi^*E \to \pi^*F$$

which is an isomorphism on $T^*M \setminus M$. We claim: This gives an element of relative K-theory $\varphi \in K^0(T^*M, T^*M \setminus M)$

$$= \tilde{K}^0(C(T^*M\setminus M \hookrightarrow T^*M))$$

If I put a Riemannian metric on M

$$= K^0(D_{T^*M}, S_{T^*M}) = K^0(T^*M^0, *)$$

1-point compactification

unit disk bundle

unit sphere bundle

$$\| \quad \tilde{K}^0(T^*M^0)$$
$$\| \text{ def} \quad K^0_c(T^*M)$$

M compact: $T^*M^0 = M^{T^*M}$

Thom space

X locally compact: $K^n_c(X) = \operatorname{colim}_{Z\subset X \text{ compact}} K^n(X, X\setminus Z)$

To define the topological index: There exists an embedding

$M \subset \mathbb{R}^N$.

$T^*M \subset \underbrace{T^*\mathbb{R}^N}_{= \mathbb{C}^N}$

The normal bundle ν is complex.
We have a tubular neighborhood U of T^*M in $T^*\mathbb{R}^N$

$(U, U \setminus T^*M) \cong (\underbrace{E\nu}_{\text{total space of the normal bundle.}}, E\nu \setminus T^*M)$

complex bundle

So we discussed the Thom isomorphism:

$$K^0(T^*M) \longrightarrow K^0(E\nu, E\nu \setminus T^*M)$$

We also have a version with compact supports

Recall symbol $\sigma(d) \in K^0_c(T^*M) \longrightarrow \underbrace{K^0_c(E\nu, E\nu \setminus T^*M)}_{\text{restrict}}$

$\overset{\cup}{Z}$ compact

$T^*M, T^*M \setminus Z$

take colimit

$$K^0_c(U, U \setminus T^*M) = K^0_c(T^*\mathbb{R}^N, T^*\mathbb{R}^N \setminus T^*M)$$

$$\downarrow$$

$$K^0_c(T^*\mathbb{R}^N) = \mathbb{Z}$$

$\mathrm{Index}_t(d)$ = the image of $\sigma(d)$ under the above map.

$\mathrm{Index}_a(d) = \dim \mathrm{Ker}(d) - \dim \mathrm{Coker}(d)$

on $\Gamma^\infty E \to \Gamma^\infty F$ smooth solution

We can find a Sobolev norm / inner product $\rightsquigarrow$ Hilbert spaces, Fredholm operator (weak solution) $\Rightarrow$ smooth

<u>Index Theorem</u>: $\mathrm{Index}_t(d) = \mathrm{Index}_a(d)$.

<u>Proof</u>: Axiomatic characterization of $\mathrm{Index}_t(d)$, satisfied by $\mathrm{Index}_a(d)$.

Where do the Todd class and the Chern character come from?

$$\mathrm{ch}(\xi) = \sum_{n \in \mathbb{N}_0} \frac{1}{n!} \psi^n(\xi)$$

power if ξ is a line bundle

recall that $S_{\mathbb{Q}} = M\mathbb{Q} \sim H\mathbb{Q}$

S (sphere spectrum) has no higher infinite homotopy groups.

$$K_{\mathbb{Q}} = K \wedge M\mathbb{Q} \overset{\sim}{=} \bigvee_{k \in \mathbb{Z}} H\mathbb{Q}[2k]$$

The Todd class: The pushforward (= Thom isomorphism) also depends on the generalized (co)homology theory. If we want to do it in ordinary (co)homology, we need to correct by multiplying by Todd class. An algebraic version - Riemann-Roch Thm.

Fulton - MacPherson: Chow groups, also singular cells in homology

groups of algebraic cycles

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Atiyah noticed : Everything in K-theory works with an action of a compact Lie group G. (in topology, we say equivariantly)

QFT : covariantly

- G-equivariant vector bundles
- equivariant Bott periodicity

$\Big\}$ $\rightsquigarrow$ equivariant homotopy theory

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