Equivariant topology

Let G be a compact Lie group. (e.g. a finite group)

The setting: A G-space X : $G \times X \to X$ associative unital.

G-CW-complex :

G-space Z $\quad Z = \bigcup_{n \geq -1} Z_n \qquad Z_{-1} \subseteq Z_0 \subseteq Z_1 \subseteq \cdots$

$Z_{-1} = \emptyset$

I_n = "set of n-cells"

$i \in I_n$: $H_i \subseteq G$ closed subgroup

isotropy

equivariant

$$\begin{tikzcd} \coprod_{i \in I_n} G/H_i \times S^{n-1} \arrow[r, "\varphi_n"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & Z_{n-1} \arrow[d] \\ \coprod_{i \in I_n} G/H_i \times D^n \arrow[r] & Z_n . \end{tikzcd}$$

G finite: Alternatively, same definition as a non-equivariant CW-cx., but I_n is a G-set, φ_n is G-equivariant.

CW-complex where G act by cell maps, isotropy constant on open cells

A G-weak equivalence of spaces: A G-equivariant map $f: X \to Y$ such that for $H \subseteq G$ closed, $f^H: X^H \to Y^H$ is a weak equivalence.

the category of G-spaces and G-homotopy classes of maps

<u>Whitehead Theorem</u>: For G compact Lie, hG-spaces have localization in G-CW-complexes with respect to weak equivalence.

$h: X \times \underbrace{[0,1]}_{G\text{ fixed}} \to Y$

① For a G-space X there exists a G-CW-complex X' and a G-weak equivalence $\gamma_X: X' \xrightarrow{\sim} X$

② If Z is a G-CW complex and $e: X \xrightarrow{\sim} Y$ is a G-weak equivalence then $[Z,X] \xrightarrow[{[Z,e]}]{\cong} [Z,Y]$

G-homotopy classes of maps

<u>G-vector bundles</u> (say, complex): For a G-space X, a (say, complex) vector bundle on X $\xi: E \to X$

total space

A G-action on E, fiberwise linear, commuting with ξ.

universal ab. group on a commutative monoid

<u>Atiyah</u>: For a compact G-CW complex X,

$$K^0_G(X) = K(\{\cong \text{ classes of } G\text{-complex vector bundles on } X\}, \oplus)$$

∋ virtual bundles

<u>Equivariant Bott periodicity</u>: $\tilde{K}^0_G(X^\xi) \cong K^0_G(X)$

say, define this as subgroup of virtual bundles of dim. 0.

Thom space of a G-vector bundle on X

In particular, a "trivial bundle": $X \times \underset{\smile}{V}$

G-space

a complex f.d. G-representation

Thom space: $\underbrace{S^V}_{}\wedge X_+$

disjoint basepoint

1-point compactification of V

So in particular, we have

$$\tilde{K}^0(S^V \wedge X_+) \cong K^0(X)$$

Briefly outlining the proof of Atiyah, only index is relevant:

G not abelian, $\dim V > 1$ (not all irreducible representations are 1-dimension)

The Bott class: $\xi : E \to X$ complex G-bundle $\dim_{\mathbb{C}} \xi = n$

$\tilde{K}^0_G(X^\xi) = K^0_G(E, E \setminus X)$

a finite chain complex of vector bundles on E which has 0 homology when restricted to $E \setminus X$.

Bott class:

$$\lambda_E : \Lambda^0_X E \xrightarrow{\wedge u} \Lambda^1_X E \xrightarrow{\wedge u} \cdots\cdots \xrightarrow{\wedge u} \Lambda^n_X E$$

fiberwise $u \in E$

The Bott class: λ_E^* $\leftarrow \mathrm{Hom}(?, \mathbb{C})$

$$\Lambda \mathbb{C}^n = \Lambda u_1 \otimes \cdots \otimes \Lambda u_n$$

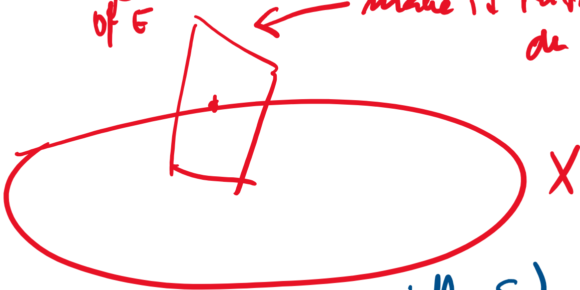
$u_1, \ldots u_n$ $\quad \mathbb{C} \xrightarrow{u_i} \mathbb{C}$

Atiyah's periodicity thm: $? \otimes \lambda_E^{\gamma} : K_G^0(X) \xrightarrow{\cong} K_G^0(E, E \setminus X)$.

To prove this, he constructs an inverse. He also considers the fiberwise projective space: $P(E \oplus 1)$ {lines through the origin in fibers of E}

fiber of E

make it into a projective space by adding an (n-1)-dimensional projective space at ∞.



Atiyah: periodicity..

$$K_G^0(E, E \setminus X) \xrightarrow{(\text{secretly: } \subseteq)} K_G^0(P(E \oplus 1))$$

$\cong K_G^0(P(E \oplus 1), P(E))$ restriction

just a virtual bundle, not a chain complex

$$\lambda_E^+ \longmapsto \sum_{k=1}^{n} (-1)^k \gamma^k \otimes \Lambda^k(E)$$

universal line bundle: over a line in a fiber of $E \oplus 1$ just take the line.

fiberwise

$$\bar{\partial} : \Omega^{0,l} \longrightarrow \Omega^{0,l+1}$$

$\Omega^{k,l}$ are (k,l)-forms

given a bundle on $P(E \oplus 1)$,

Atiyah defines a v. G-bundle on X, takes the index of this operator fiber-wise.