Math 696

Friday, March 21, 2025 8:57 AM

Atiyah: G-equivariant Bott periodicity. G = compact Lie group,

X compact G-space $\xi : E \to X$ $\underbrace{G\text{-equivariant complex vector bundle.}}$

complex bundle on X with G-action compatible with projection

? Bott element $\lambda_E^* \in K_G^0(E, E \setminus X)$

$$\Lambda^0 \xi^* E \xrightarrow{\wedge u} \Lambda^1 \xi^* E \xrightarrow{\wedge u} \cdots \xrightarrow{\wedge u} \Lambda^k \xi^* E$$

dim E

given element of the fiber

(fiber-wise)

exterior powers of the pullback of the vector bundle E to the total space of E by projection

finite chain complex of vector bundles on E, has homology 0 on $E \setminus X$

$$K_G^0(E, E\setminus X) \underset{\text{excision}}{\cong} K_G^0(P(E\oplus 1), \underbrace{P(E)}_{P(E\oplus 1)\setminus X}) \xrightarrow{\text{extend}} K_G^0(P(E\oplus 1))$$

$$\lambda_E^* \longmapsto \sum_{i=0}^{k} (-1)^i \gamma^i \Lambda^i(\xi^* E)$$

Atiyah: describes an inverse of that map ?. $\lambda_E^* : K_G^0(X) \to K_G^0(E, E\setminus X)$

$\alpha : K_G^0(P(E\oplus 1)) \longrightarrow K_G^0(X)$

Briefly recall (k,l)-forms: If M is a complex manifold, locally holomorphically diffeomorphic to an open set in $\mathbb{C}^n$.

$$T_M \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}\left\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \,\middle|\, i=1,\dots,n\right\}$$

on $\mathbb{C}$ if $z = x+iy$
$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Dual basis of
$$\underset{\substack{\| \\ \Omega^1(M;\mathbb{C})}}{T_M^* \otimes_{\mathbb{R}} \mathbb{C}} = \mathbb{C}\{\underset{\substack{\uparrow \\ \Omega^{1,0}(M;\mathbb{C})}}{dz_i}, \underset{\substack{\nwarrow \\ \Omega^{0,1}(M;\mathbb{C})}}{d\bar{z}_i} \mid i=1,\dots,n\}$$

(k,l)-forms:
$$\Omega^{k,l}(M) = \Lambda^k \Omega^{1,0}(M;\mathbb{C}) \otimes \Lambda^l \Omega^{0,1}(M;\mathbb{C}).$$

$$d = \partial + \bar{\partial}, \quad \partial : \Omega^{k,l}(M) \to \Omega^{k+1,l}(M)$$
$$\bar{\partial} : \Omega^{k,l}(M) \to \Omega^{k,l+1}(M)$$
$$\bar{\partial}(h\, \underbrace{dz_{i_1} \wedge \cdots dz_{i_k} \wedge d\bar{z}_{j_1} \wedge \cdots d\bar{z}_{j_l}}) =$$
$$= \sum_j \frac{\partial}{\partial \bar{z}_j}\, d\bar{z}_j \wedge \cdots$$

Suppose I have a bundle F on $P(E\oplus 1)$

$$\bar\partial_F : \bigoplus_{k\,\text{even}} \Omega^{0,k}(P(E\oplus 1))\otimes F \longrightarrow \bigoplus_{k\,\text{odd}} \Omega^{0,k}(P(E\oplus 1))\otimes F.$$

fiberwise

The map $\alpha = \operatorname{index} \bar\partial_F = \underbrace{\operatorname{Ker}\bar\partial_F - \operatorname{Coker}\bar\partial_F}_{\text{fiberwise.}}$

Computation $\sum (-1)^k \gamma^k \underbrace{\Lambda^k(\xi^* E)}$

index $= 0$ $k>0$

1 $k=0$

This shows α is left inverse to $\cdot\,\lambda^*_E$. A part of the proof of the index thm. shows that it is also a right inverse.

The group G plays no role!

But if we want to write down a modern generalized cohomology theory, can we classify G-equivariant complex n-bundles?

What is a G-equivariant generalized cohomology theory?

$G = \{e\}$

Recall that non-equivariant complex n-bundles are classified by $BU(n)$ $GL_n(\mathbb{C})$

Grassmannian model: n-dimensional $\mathbb{C}$-vector subspaces $V \subset \bigoplus_{\infty} \mathbb{C}$.

X paracompact $\Rightarrow$ $[X, BU(n)] \cong \{\cong$ classes of complex n-bundles on $X\}$

homotopy classes $(f : X \to BU(n)) \longmapsto f^* \gamma^n_{\mathbb{C}}$.

$\gamma^n_{\mathbb{C}}$ over V has the fiber V

To work G-equivariantly, replace $\bigoplus_{\infty} \mathbb{C} = \mathcal{U} = \bigoplus_{\infty} \bigoplus V$

complete universe (May)

all irreducible complex representations of G

We can put $BU(n)_G := \{ n\text{-dim. cx. vector subspaces } V \subset \mathcal{U} \}$

(All irreducible rep. of a compact topological group are finite-dim.)

Th. If X is a paracompact G-space, then

$$[X, BU(n)_G] \cong \{ \cong \text{classes of } G\text{-equivariant complex } n\text{-bundles on } X \}$$

Note: $BU(n)_G^H \overset{\text{non-eq.}}{\simeq} \coprod_{n_1+\dots+n_k=n} BU(n_1) \times \dots \times BU(n_k)$ (?)

H-fixed points

irreducible H-rep. of dim. n_i

Exercise: Work that out.

We want a more "explicit" description of $BU(n)_G$ from the point of view of homotopy theory.

The important property of $BU(n)$ is that we have a fibration sequence

$$U(n) \to EU(n) \to BU(n)$$

$EU(n)$: orthonormal n-frames in $\bigoplus_{\infty} \mathbb{C}$

$EU(n) \simeq *$

For a compact Lie group H, we always have a fibration sequence

$$H \to EH \to BH$$

EH: total space of universal principal H-bundle; <u>contractible</u> free H-CW-complex.

non-equivariant picture

I will now describe $B_G H$ for two compact Lie groups H, G, classifying G-equivariant principle H-bundles.

A $\underbrace{\text{family}}_{\text{Map}}$ $\mathcal{F}$ of subgroups of a compact Lie group Γ is a

contains conjugacy and subgroups

set of subgroups of Γ which is closed under subconjugacy:

$$K \in \mathcal{F},\ g \in \Gamma,\ \underset{\text{subgroup}}{K' \subseteq G},\ g^{-1}K'g \subseteq K \Rightarrow K' \in \mathcal{F}.$$

To classify G-equivariant principal H-bundles, consider the family $\mathcal{F}_G H$ of subgroups $K \subseteq G \times H$ where $K \cap (\{e\} \times H) = \{e\}$.

For any family $\mathcal{F}$ of subgroups of a compact Lie group Γ, there exists a G-CW-complex $E\mathcal{F}$ such that

$$E\mathcal{F}^K \simeq * \quad \text{when } K \in \mathcal{F}$$

means K-fixed points

$$E\mathcal{F}^K = \emptyset \quad \text{when } K \notin \mathcal{F}.$$

Classification of G-equivariant principal H-bundles:

$$\boxed{B_G H = \underbrace{E\mathcal{F}_G H}_{H\text{-free}} / H}$$