Classification of G-equivariant bundles (G compact Lie group)

① H-principal bundles (H another compact Lie group)

Recall: Γ compact Lie group: A <u>family</u> $\mathcal{F}$ is a set of closed subgroups of Γ closed under subconjugacy:

$$K \in \mathcal{F} \ \& \ g^{-1}K'g \subseteq K, g \in \Gamma \Rightarrow K' \in \mathcal{F}.$$

For a family $\mathcal{F}$, there exist a G-CW-complex $E\mathcal{F}$

fixed point

$$E\mathcal{F}^K = \emptyset \qquad K \notin \mathcal{F}$$

$$E\mathcal{F}^K \simeq * \qquad K \in \mathcal{F}$$

The recipe for the classifying space of G-equivariant principal H-bundles:

$$\Gamma = G \times H \qquad \mathcal{F} = \{ K \subseteq G \times H \mid K \cap H = \{e\} \}$$

family in Γ

$$\boxed{B_G H = E\mathcal{F}/H}$$

say, G-CW complex

Theorem: X is a $\overbrace{\text{paracompact } G\text{-space}}^{\text{say, } G\text{-CW complex}}$ then

$$\{\cong \text{classes of } \underbrace{G\text{-equivariant principal } H\text{-bundles}}_{\substack{\text{principal } H\text{-bundle with} \\ \text{a } G\text{-action preserving the structure}}}\} \cong [X, B_G H]$$

$f^* \gamma$ ⟵ $(f : X \to B_G H)$

$[X, B_G H]$: unbased G-homotopy classes of G-equivariant maps

The universal G-equivariant principal bundle on $B_G H$:

$B_G H = E\mathcal{F}/H$

<u>Key step of the proof:</u> $E\mathcal{F} \xrightarrow{\gamma} B_G H$

If X is G-CW complex where $X^K = \emptyset$ for $\underbrace{K \notin \mathcal{F}}_{K \cap H \neq \{e\}}$

Then there exist a <u>unique, up to G-homotopy</u> G-map $X \to E\mathcal{F}$. } Whitehead thm. inductions by cells

instead of $G \times H$

<u>Note:</u> There is a twisted version: $1 \to H \to \Gamma \to G \to 1$

In particular for (say, complex) $\underbrace{G\text{-equivariant vector bundles}}_{\substack{\text{associated bundles to } G\text{-equivariant} \\ \text{principal } U(n)\text{-bundles}}}$ G-equiv. Hermitian metric

Grassmannian model of $B_G U(n) = \{n\text{-dim. vector subspaces of } \mathcal{U} = \bigoplus_{\substack{\text{irreducible} \\ G\text{-rep. } V}} \bigoplus_{\infty} V\}$

universe

We can check that this is a model of the space defined above $(H = U(n))$:

$$E_G U(n) = \{\text{orthonormal } n\text{-frames in } \mathcal{U}\}$$

$$\text{|| ?}$$

$$E\mathcal{F}$$

bijectively correspond to $G' \leq G,\ h: G' \to H$

$$\mathcal{F} = \{K \leq G \times H \mid K \cap H = \{e\}\}$$

$G' = \mathrm{Im}\ \mathrm{proj}_1|K$

$$\begin{tikzcd} K \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\mathrm{proj}_1", "\cong"'] & H \\ G' & \end{tikzcd}$$

homomorphism: $G' \to H$

$\mathrm{proj}_1 : G \times H \to G$

$K \in \mathcal{F} \Rightarrow \mathrm{proj}_1|_K : K \underset{\leq}{\to} G$ is injective $(k \in \mathrm{Ker}(\mathrm{proj}_1|_K) \Rightarrow k \in H)$

In the case of $H = U(n)$: $G' \subseteq G$, $\underbrace{h : G' \to U(n)}_{\text{n-dimensional (unitary) representation of } G'.}$

Putting $E_G U(n) = \{ n\text{-}\overset{\text{unitary}}{\text{frames}} \text{ in } \mathcal{U} \}$

complete universe

$E_G U(n)^K = \emptyset$ $\qquad K \cap U(n) \neq \{e\}$

$K \cap U(n) = \{e\}$: $G' \subseteq G$ $\underbrace{\text{and an } n\text{-dim. rep of } G'}_{\text{product of irreducible } G'\text{ rep.}}$

$V_1 \oplus \cdots \oplus V_n$

take orthonormal bases

$$E_G U(n)^K = \operatorname{Hom}_{G'}(V_1 \oplus \cdots \oplus V_n, \mathcal{U}) = \prod_{i=1}^{n} \operatorname{Hom}(V_i, \mathcal{U}) \quad \bigoplus_{\infty} V_i \oplus \underbrace{\bigoplus \text{other irreducible rep.}}_{\text{do not matter by Schur lemma}}$$

$$= \prod_{i=1}^{n} \underbrace{\operatorname{Hom}\left(V_i, \bigoplus_{\infty} V_i\right)}_{\operatorname{Hom}(\mathbb{C}, \bigoplus_{\infty} \mathbb{C}) \simeq *} = *$$

What does Atiyah's equivariant Bott periodicity really say?

<u>Theorem</u>: Let V be a finite dimensional complex G-representation (G compact Lie group). Then

$$\underbrace{\Omega^V (B_G U \times \mathbb{Z})}_{\mathrm{Map}(S^V,\, B_G U \times \mathbb{Z})} \simeq B_G U \times \mathbb{Z}$$

$\simeq$: G-homotopy equivalence

S^V : 1-point compactification of V.

Map: all maps, not necessarily equivariant, with G-action by conjugation $g(f)(x) = g f(g^{-1}(x))$

A part of the story: for X a compact G-CW-complex,

$$K_G^0 X = [X, B_G U \times \mathbb{Z}]^G$$

$[\ ,\ ]^G$ ← homotopy classes of G-maps.

$$= K(\{\cong \text{ classes of } G\text{-bundles on } X\}, \oplus)$$

K: univ. abelian group on a commutative monoid

$\{\cong$ classes$\}$: commutative monoid

G-bundles: unspecified their dimension, locally constant

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This leads to the concept of G-equivariant generalised cohomology theories, and G-equivariant spectra.

$\underbrace{\text{G-equivariant cohomology}}_{RO(G)\text{- graded}}$: $\tilde{E}^V_G(X)$ abelian group when X is a based G-CW-complex, $V \in RO(G)$

The real representation ring $= \mathbb{Z}\{\text{irreducible real representations of } G\}$

virtual representation

$\mathbb{Z} \subseteq RO(G)$

trivial real representations

Axioms: ① $f : X \to Y$ G-equivariant map $\Rightarrow$

$$\tilde{E}^V_G Cf \xrightarrow{i^*} \tilde{E}^V_G Y \xrightarrow{f^*} \tilde{E}^V_G X$$ is exact

$\tilde{E}^{V-1}_G X$

(as a consequence, long exact sequence) $\tilde{E}^{V+1}_G Cf \to \cdots$

② We are given, for W a f.d. real G-representation natural isomorphisms

$$\tilde{E}^{V\oplus W}_G(\underbrace{\Sigma^W X}_{S^W \wedge X}) \cong \tilde{E}^V_G(X).$$

Example: $\tilde{K}^V_G$ (Atiyah)