G-equivariant spectra

$\rightsquigarrow$ representing genuine equivariant (co)homology theories, G compact Lie

RO(G)-grade

$\mathcal{U}$ = complete universe = $\bigoplus\limits_{\infty} \bigoplus\limits_{\substack{V \\ \text{irreducible} \\ G\text{-rep.}}} V$ with a G-invariant inner product

$n \rightsquigarrow$ trivial n-dim. real representation

$V \underbrace{\subset\subset}_{\text{finite-dim. real subrepresentation}} \mathcal{U} \longmapsto$ based G-space Z_V

$V \subseteq W \subset\subset \mathcal{U} \longmapsto \sigma_{V,W} : Z_V \xrightarrow{\cong} \underbrace{\Omega^{W-V} Z_W}_{F(S^{W-V}, Z_W)}$

G-equivariant homeomorphism

orthogonal complement

based continuous maps, G-action by conjugation

G-spectrum indexed over $\mathcal{U}$

Axioms:

$$\begin{tikzcd}
Z \arrow[r, "\sigma_{V,W}"] \arrow[d, "\sigma_{V,U}"'] & \Omega^{W-V} Z_W \arrow[d, "\Omega^{W-V}\sigma_{W,U}"] \\
\Omega^{U-V} Z_U \arrow[r, "="] & \Omega^{W-V}\Omega^{U-W} Z_U
\end{tikzcd}$$

$V \subseteq W \subseteq U \subset\subset \mathcal{U}$

$\sigma_{V,V} = \mathrm{Id}.$

morphisms $f: (Z_V)_V \to (T_V)_V$ of G-$\mathcal{U}$-spectra

tuples $f_V : Z_V \to T_V$ preserving the structure

The theory is developed similarly as non-equivariantly, with some changes (to be pointed out).

- prespectra - same definition, drop requirement that $\tilde{\sigma}_{V,W}$ be homeomorphisms (different categories for different cofinal systems of eq. $\subset\subset \mathcal{U}$)
- forget : $\mathcal{U}$-Spectra $\to$ $\mathcal{U}, \{V\}$-prespectra has a left adjoint L (spectrification) due to Freyd-Kelly. reference: LNM 1213
- (continuous) limits in $\mathcal{U}$-spectra $\leftarrow$ level-wise limits
  colimits $\leftarrow$ level-wise in prespectra, spectrify
- X based G-space ; Z $\mathcal{U}$-spectrum $\rightsquigarrow$ $X \wedge Z$, $F(X,Z)$
  cont. colimit, cont. limit
- homotopy of $\mathcal{U}$-spectra: $[0,1]_+ \wedge Z \longrightarrow T$.

  Suspension spectrum: Σ^∞ : based G-spaces $\to$ $\mathcal{U}$-spectra left adjoint to $(Z_V) \longmapsto Z_0$.
- shift desuspensions by $W \subset\subset \mathcal{U}$: $Z = (Z_V)$

$$Z[-W]_V = \begin{cases} Z_{V-W} & \text{if } W \subseteq V \\ * & \text{else} \end{cases} \quad \text{spectrify}$$

orthogonal complement

- Spheres S^α, $\alpha \in RO(G)$: $\alpha = U - W$ formal subtraction

  f.d. G-rep.

$$S^\alpha := \Sigma^\infty S^U[-W]$$ $\longleftarrow$ up to isomorphism, does not depend on choice

We could define $\pi_\alpha Z = [S^\alpha, Z]$, $\alpha \in RO(G)$

$\mathcal{U}$-spectrum

homotopy classes of morphisms

In fact, for $n \in \mathbb{Z}$, we define

$$\underbrace{\pi_n^H(Z) = [G/H_+ \wedge S^n, Z]}$$

Z_H = forget to H-spectrum, $H \subseteq G$ closed subgroup

$$= [S^n, Z_H]$$

H-$\mathcal{U}$-spectra homotopy classes of morphisms

__A weak equivalence__ : $f: Z \to T$ is a morphism of $\mathcal{U}$-spectra which induces $\cong$ on π_n^H for all $H \subseteq G$ closed, $n \in \mathbb{Z}$.

Cell G-spectra: Z

$$* = Z_{(-1)} \longrightarrow Z_{(0)} \longrightarrow Z_{(1)} \longrightarrow \cdots \qquad Z = \operatorname{colim}(Z_{(n)})$$

I_n = set of n-cells $\qquad d_n : I_n \to \mathbb{Z}$

$H_i \leq G$ closed, $i \in I_n$

$$C\Big(\bigvee_{i \in I_n} S^{d_n(i)-1} \wedge G/H_i \xrightarrow{\varphi_{(n)}} Z_{(n-1)}\Big) = Z_{(n)}$$

mapping cone

Attaching map

complete universe

Theorem (Lewis–May): hG-U-spectra has colocalization with respect to G cell spectra and weak equivalences.

objects, homotopy classes of morphisms

the Whitehead theorem for equivariant spectra

Stability: Recall that $\Omega^V Z$, $Z[-V]$ for $\mathcal{U}$-spectrum Z, $V \subset \mathcal{U}$

are not (known to be) isomorphic.

Stability theorem (a.k.a. we don't care about this in derived categories)

preserve equivalences

In D-spectra, $D\Omega^V$, $D?[-V]$ are isomorphic functors, which are inverse to $(L\Sigma)^V ?$.

inverse equivalences of categories DG-spectra $\to DG$-spectra.

If X is a based G-CW-complex and E is a G-$\mathcal{U}$-spectrum, then we have $RO(G)$-graded generalized G-equivariant homology and cohomology based on E:

morphisms in DG-spectra

$$\tilde{E}_\alpha X := [S^\alpha, X \wedge E]$$

$$\tilde{E}^\alpha X := [S^{-\alpha}, F(X, E)]$$

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Note: A key feature is duality. A finite G-CW complex X embeds into a f.d. G-representation. (Exercise: Why?)

$\Rightarrow$ Spanier-Whitehead duality

E G-spectrum, then $\underbrace{(\pi_n^H E)_{\substack{H \subseteq G \\ \text{closed}}}}_{\text{Mackey functor } \underline{\pi}_n E.}$, n fixed (to be explained)