Math 696

Friday, March 28, 2025 9:00 AM

A compact G-CW-complex X embeds into a finite-dimensional G-representation

(G compact Lie group).

$$C(X) = \mathbb{C}X$$

free $\mathbb{C}$-vector space on X

finite linear combinations of points in X

$$\Big\langle \sum_{x\in X} a_x x \,,\, \sum_{x\in X} b_{x'} x \Big\rangle = \sum_{x\in X} a_x \bar{b}_x$$

G-invariant inner product

finite sums

completion w.r.t. this

$\widehat{\mathbb{C}X}$ = Hilbert space G acts on it unitarily.

Peter-Weyl theorem: $\widehat{\mathbb{C}X} = \widehat{\bigoplus}\, \underbrace{\text{f.d. G-rep.}}_{\text{unitary}}$

Hilbert

X

System of maps into f.d. subrepresentations, one of them injective by compactness complete universe

The derived category DG-spectra has Spanier-Whitehead duality:

finitely many cells — compact Lie

X a finite G-CW-complex, $DX_+ = \Sigma^{-V} C_j$.

we have a spectrum $S^{-V} = S[-V]$!

$X_+ \overset{i}{\subset} S^V$ ← f.d. G-representation

$j : S^V \setminus X_+ \hookrightarrow S^V$

$X \subset V$

C_j " $= S^V / S^V \setminus X_+$ "

this is not literally true because j is not a cofibration

mapping cone = the derived form of the quotient, which is what we want.

This is what makes, for a based G-CW-complex X, and a G-spectrum E, $V \in RO(G)$,

$$\tilde{E}_V X := [S^V, E \wedge X]$$

← morphisms in DG-spectra

into a G-equivariant generalised homology theory. Cohomology:

$$\tilde{E}^V X := [S^{-V}, F(X, E)].$$

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Also defining E-homology$^{(\omega)}$ for arbitrary G-spectra E, X:

The smash product of spectra: If X, Y are G-$\mathcal{U}$-spectra (complete universe)

$X \wedge Y$ G-$\mathcal{U}\oplus\mathcal{U}$-spectrum

$$Z_{\underbrace{V\oplus W}} := X_V \wedge Y_W$$

$\mathcal{U}\oplus\mathcal{U}$-prespectrum

defined on the $\mathcal{U}\oplus\mathcal{U}$-cofinal set of these f.d. subrepresentations

$$\begin{array}{ccc} V & \oplus & W \subset\subset \mathcal{U}\oplus\mathcal{U} \\ \cap & & \cap \\ \mathcal{U} & & \mathcal{U} \end{array}$$

$$X \wedge Y := L(Z_{V\oplus W})$$

spectrification

This behaves well if X, Y are G-cell spectra. To define $X \wedge Y$ as a G-$\mathcal{U}$-spectrum, push it forward $i_\#$ via an isometry $i: \mathcal{U}\oplus\mathcal{U} \hookrightarrow \mathcal{U}$.

isometry of universes

$i: \mathcal{V} \to \mathcal{U}$

$i_\#$ left adjoint to i^*

$i^*: \mathcal{U}$-spectra $\to$ $\mathcal{V}$-spectra

$X \qquad (i^* X)_V := X_{i(V)}$

$$\mathrm{Iso}(\mathcal{U}\oplus\mathcal{U}, \mathcal{U}) \simeq *$$

For E, X cell G-spectra, define

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$$E_V X = [S^V, E \wedge X] \leftarrow \text{morphisms in } DG\text{-spectra}$$

$$E^V X = [S^{-V}, F(X,E)],$$

$F(X, ?)$ right adjoint to $X \wedge ?$

$V \in RO(G)$.

How do we calculate in G-spectra?

Let's go back to the non-equivariant case: How did we calculate in DSpectra?
(e.g. cobordism)

We focused first on spectra representing ordinary (co)homology theories.

Non-equivariantly: A abelian group, HA

singular (co)homology $\begin{cases} H_n(?;A) \\ H^n(?;A) \end{cases}$

But what is "ordinary" (co)homology equivariantly?

"Ordinary" (co)homology means satisfying the dimension axiom.

$$E_n(*) = 0 \quad \text{for } n \neq 0.$$
$$\parallel$$
$$E^{-n}(*)$$

Equivariantly, this axiom is the same $\boxed{\text{for } n \in \mathbb{Z}}$!

(too strong for $n \in RO(G)$)

would trivialize things.

we will calculate examples.

$\mathbb{Z}/2$

Let G be <u>finite</u>. E is called an <u>Eilenberg-MacLane</u> G-spectrum if

$$\underbrace{E_n(G/H)}_{E^H_n(*)} = 0 \quad \text{for all } \underset{n \in \mathbb{Z}}{n \neq 0},\ H \subseteq G.$$

What can then $(E_0(G/H))_{H \subseteq G}$ look like?

(equivalently, what is the structure on $\pi_n^H(E) = E_n(G/H)$, $H \subseteq G$ for a fixed n for any spectrum E?)

$\boxed{\text{Mackey functor}}$ ← a concept that occurs in homotopy theory, representation theory, number theory

Suppose we have a G-equivariant map

$$G/H \to G/K.$$

Functoriality:

$$E_0\, G/H \xrightarrow{r} E_0\, G/K$$

Strong dual (Spanier-Whitehead)

$$DM_+ = M^{\nu}$$

← virtual normal bundle put in dim. $-\dim M$

This is how we prove Poincaré duality

Recall duality. G finite $\Rightarrow$ G/H is a 0-dim. G-manifold.

$$DG/H_+ = G/H_+$$

G finite

we get also a map in the opposite direction Dr:

$$E_0\, G/K \xrightarrow{Dr = c} E_0\, G/H$$

coinduction

So we know that a Mackey functor M will assign an abelian group $M(G/H)$ for a G-orbit G/H and for a G-map $\underline{G/H \xrightarrow{f} G/K}$, we get homomorphisms in both directions:

$$M(G/H) \xrightarrow{r} M(G/K)$$
$$M(G/K) \xrightarrow{c} M(G/H).$$

Precise axioms $\underline{\underline{\text{next time.}}}$

informed by equivariant stable homotopy theory.