Recall the $\underbrace{\text{orbit category}}_{\mathcal{O}_G}$. Let G be a finite group.

Obj $\mathcal{O}_G = \{G/H \mid H \subseteq G\}$

subconjugacies from H to K

$\mathrm{Mor}_{\mathcal{O}_G}(G/H, G/K) = \overbrace{G\text{-equivariant maps (using the left } G\text{-action)}}$ normalizer

$$\mathrm{Aut}_{\mathcal{O}_G}(G/H) = N(H)/H = \underbrace{W(H)}_{\text{Weyl group}}$$

$G/H \longrightarrow G/K$ G-equivariant?

$$\left.\begin{array}{rcl} eH & \longmapsto & gK \\ \| & & \| \\ hH & \longmapsto & hgK \end{array}\right\} \quad g^{-1}hg \in K$$

replace g by gk, $k \in K$, we get the same morphism

Answer: $\boxed{g \in G/K \text{ so that } \underbrace{g^{-1}Hg \subseteq K}_{\text{subconjugacy}}.}$

Recall that the concept of a Mackey functor is motivated by the structure on $(\pi_n(E^H))_{H \subseteq G}$, with chosen n, on a given G-spectrum E.

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$[G/H_+, E]_{DG\text{-spectra}} = [S, G/H_+ \wedge E]$

Source contravariant: $\mathcal{O}_G^{op} \to Ab$ covariant: $\mathcal{O}_G \to Ab$

on morphisms: restrictions r, coinductions c, transfers

These functors coincide on objects

$M(G/H)$

Mackey functor

Duality: $DG/H_+ = G/H_+$

strong dual in DG-spectra

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Duality gives us an axiom called the <u>projection formula</u>.

Extending a Mackey functor M to finite G-sets X: $\quad X \cong \coprod_{i=1}^{n} G/H_i$.

$$MX := \bigoplus_{i=1}^{n} M(G/H_i).$$

a G-set

$$\begin{tikzcd} G/H \times_{G/K} G/J \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G/J \arrow[d] \\ G/H \arrow[r] & G/K \end{tikzcd}$$

finite group

<u>Definition</u>: A G-<u>Mackey functor</u> is a pair of functors $\mathcal{O}_G^{op} \to Ab$, $\mathcal{O}_G \to Ab$ which coincide on objects (notation: $M(G/H)$), r on $\mathcal{O}_G^{op}$, c on $\mathcal{O}_G$, satisfying the projection formula.

<u>Projection formula</u>:

$$\begin{tikzcd} M(G/H \times_{G/K} G/J) \arrow[r, "c"] & M(G/J) \\ M(G/H) \arrow[u, "r"] \arrow[r, "c"] & M(G/K) \arrow[u, "r"] \end{tikzcd}$$

$$\coprod_{g \in H\backslash K/J'} G/(g^{-1}H'g \cap J')$$

H, J are both subconjugate to K

$H' \subseteq K$ conjugate to H

$J' \subseteq K$ conjugate to J

The double coset formula

Example: $G = \overbrace{\mathbb{Z}/p}^{C_p}$.

$$G/\underbrace{H} \quad \text{orbit}$$

isotropy

$\mathcal{O}_G$ isotropy

$$\begin{tikzcd} \mathbb{Z}/p & \bullet \arrow[loop above, "Id"] \\ \{0\} & \bullet \arrow[u] \arrow[loop below, "\mathbb{Z}/p"] \end{tikzcd}$$

only one morphism

$$\{g \in \underbrace{G/K} \mid g^{-1}Hg \subseteq K\}$$

$K = \mathbb{Z}/p$
$H = \{0\}$

$\mathbb{Z}/p$-Mackey functor M:

Isotropy

$$\begin{tikzcd} \mathbb{Z}/p & \bullet \arrow[d, bend right, "r"'] \\ \{0\} & \bullet \arrow[u, bend right, "c"'] \arrow[loop below, "\mathbb{Z}/p"] \end{tikzcd}$$

$\gamma :=$ generator of $\mathbb{Z}/p$

$$\begin{tikzcd} M(\mathbb{Z}/p/\mathbb{Z}/p) \arrow[d, phantom, "\parallel"] \\ M(\mathbb{Z}/p) \arrow[d, bend right, "r"'] \\ M(\{0\}) \arrow[u, bend right, "c"'] \arrow[d, phantom, "\parallel"] \\ M(\mathbb{Z}/p/\{1\}) \end{tikzcd}$$

isotropy notation

abelian group

$\gamma : M(\{0\}) \to M(\{0\})$

$\gamma^p = Id$

$\left.\begin{array}{l} \gamma r = r \\ c\gamma = c \end{array}\right\}$ functoriality

$$\underbrace{rc = 1 + \gamma + \cdots + \gamma^{p-1}}_{\text{double coset formula}}$$

General examples of Mackey functors:

The Burnside Mackey functor

$$\mathcal{A}(H) = A(H)$$

Isotropy notation

Connection with equivariant homotopy theory:

$$\mathcal{A} = \underline{\pi}_0(S).$$

Burnside ring for G a finite group:

X

$K(\{\cong \text{ classes of finite } G\text{-sets}\}, \amalg)$

$= \mathbb{Z}\{G/H \mid (H) \subseteq G\}$

count conjugates only once

restrictions = restrict the action

corestrictions = induction

$H \subseteq K$

$H\text{-set } X \longmapsto K \times_H X$

Exercise: Mackey functor

(we will return to why it is a ring)

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The representation ring

Mackey functor $\mathcal{R}$:

$\mathcal{R}(H) = R(H)$

$\mathcal{R} = \underline{K}^0_G(*)$

$R(H) = \mathbb{K}\{\cong$ classes of f.d. cx. representations of $H, \oplus\}$ $\otimes$

$= \mathbb{Z}\{\cong$ classes of irreducible representations$\}$

restrictions = restrictions

coresfunctions = Induction

H-rep. $V \longmapsto \mathbb{C}K \otimes_{\mathbb{C}H} V$

The "constant" Mackey functor:

Let Q be an abelian group

$\underline{Q}(H) = Q$

restrictions = Id

corestriction: $H \to K$: multiplication by $[K:H] = \frac{|K|}{|H|}$

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