Math 696

Wednesday, April 2, 2025 8:58 AM

Ordinary equivariant homology and cohomology (taking into account fixed point under subgroups)

Bredon cohomology

G can be a discrete group

The orbit category $\mathcal{O}_G$

$\mathrm{Obj}(\mathcal{O}_G) = \{\text{orbit } G/H \mid H \subseteq G\}$

morphisms = G-equivariant maps (= subconjugacies)

Coefficient system: A functor $\mathcal{O}_G^{op} \to Ab$. } an abelian category, enough projectives

Given a G-space X, we can get a chain complex of coefficient systems

$C_G(X)_n : \quad G/H \longmapsto C_n(X^H)$

fixed points

singular chains $= \{ G/H \times \Delta^n \to X \}$ G-equivariant

projective coefficient systems

Given a G-coefficient system $\mathcal{U} : \mathcal{O}_G^{op} \to Ab$,

$$H^n(X; \mathcal{U}) := H^n \underbrace{\mathrm{Hom}_{\mathcal{O}_G^{op}}(C_G(X), \mathcal{U})}$$

in the abelian category of coefficient systems

cochain complex of ab. groups

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If G is a finite group, we can consider the special case when $\mathcal{G}$ is a $\underbrace{\text{Mackey functor}}_{\text{coefficient system}}$. $H^n(X;\mathcal{G})$ already defined.

But now we also have homology:

$\mathcal{G} : \mathcal{O}_G \to Ab$

$$H_n(X;\mathcal{G}) := H_n\Big(\mathcal{G} \otimes_{\mathcal{O}_G^{op}} C_G(X)\Big)$$

colimit $\left(\bigoplus_{\varphi : G/H \to G/K} \mathcal{G}(G/H) \otimes C_n(X^K) \rightrightarrows \bigoplus_{H \subseteq G} \mathcal{G}(G/H) \otimes \underbrace{C_G(X)_n(G/H)}_{C_n(X^H)} \right)$

@ given n;

Preview: $\underbrace{H_n(?;\mathcal{G}),\ H^n(?;\mathcal{G})}_{\text{singular homology \& cohomology w. coeffs. in a Mackey functor}}$ } are represented by G-spectrum $H\mathcal{G}$ ← Mackey functor

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Note: X: G-CW-complex, set of n-cells = G-set } $C_G^{cell}(X)_n = \{ I_n = \text{set of } n\text{-cells} \}$

$C_G(X)_n(G/H) = I_n^H$

$H^n_G(X;\mathcal{G}) = H^n_G(X;\mathcal{G})_{cell}$

G finite & Mackey functor ⇒ same in homology

generalised (co)homology: Atiyah-Hirzebruch spectral sequence

Back to Mackey functors (G finite group)

A different description: A span of finite G-sets is a diagram of G-sets of the form

$$X \longleftarrow Y \longrightarrow Z \quad /\sim$$

finite G-sets

G-maps

equivalence relation

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] \arrow[dd, "\cong"] & \\ X & & Z \\ & Y' \arrow[ul] \arrow[ur] & \end{tikzcd} \quad \sim$$

Spans of finite G-sets form a category $\mathcal{B}^+$

composition:

$$\begin{tikzcd} & & Y \times_Z T \arrow[dl] \arrow[dr] & & \\ X & Y \arrow[l] \arrow[r] & Z & T \arrow[l] \arrow[r] & U \end{tikzcd}$$

$X \leftarrow Y_1 \rightarrow Z \oplus X \leftarrow Y_2 \rightarrow Z$

$= X \leftarrow Y_1 \sqcup Y_2 \rightarrow Z$

Operations on spans: $\sqcup$, $(\times)$.

on morphisms

almost an abelian category (no inverse)

Can formally introduce an inverse

$\mathrm{Hom}_{\text{spans}}(X, Z)$ commutative monoid

an abelian category: The Burnside category

$$\mathcal{B}^+ \rightsquigarrow \mathcal{B}$$

Apply K (univ. abelian group)

A Mackey functor = An additive functor $\mathcal{B} \to Ab$

equivalent data

$\mathcal{B}^+ \to Ab$

$X \leftarrow Y \longrightarrow Z$

$G/H \xrightarrow{f} G/K \longmapsto : M(X) \xrightarrow{r} M(Y) \xrightarrow{c} M(Z)$

$M(G/K) \xrightarrow{r} M(G/H) \longleftarrow$

$G/K \xleftarrow{f} G/H \xrightarrow{Id} G/H$

$M(G/H) \xrightarrow{c} M(G/K)$

$G/H \xleftarrow{Id} G/H \xrightarrow{f} G/K$

The Burnside category point of view explains how Mackey functors form a tensor category

M, N Mackey functors $\quad M, N : \mathcal{B} \longrightarrow Ab$

$M \square N$

$$\begin{tikzcd}
\mathcal{B} \boxtimes \mathcal{B} \arrow[r, "M \boxtimes N"] \arrow[d, "X"'] & Ab \arrow[d, equal] \\
\mathcal{B} \arrow[r, dotted, "M \square N"'] & Ab
\end{tikzcd}$$

product of spans X