Math 696

Friday, April 4, 2025 8:57 AM

<u>Burnside category</u> : $\mathcal{B} = \mathcal{B}_G$ is an abelian category

finite group

Mackey functors = Additive functors $\mathcal{B} \to Ab$.

$\text{Obj}\,\mathcal{B}$ = finite G-sets $\oplus = \amalg$

$$\text{Mor}_{\mathcal{B}}(G/H, G/J) = \mathbb{Z}\{G/H \leftarrow G/K \rightarrow G/J\}$$

$$= \mathbb{Z}\{(g \in H\backslash G/J,\ K \subseteq H) \mid g^{-1}Kg \subseteq J\}$$

To describe the tensor product of Mackey functors: $\square$. (Dress)

$\mathcal{B} \boxtimes \mathcal{B}$

$$\text{Obj}\,\mathcal{B}\boxtimes\mathcal{B} = \text{“}\coprod_{H_1,H_2 \subset G}\text{”} \begin{pmatrix} G/H_1 \\ G/H_2 \end{pmatrix}$$

$\oplus$

$$\text{Mor}_{\mathcal{B}\boxtimes\mathcal{B}}\left(\begin{pmatrix} G/H_1 \\ G/H_2 \end{pmatrix}, \begin{pmatrix} G/J_1 \\ G/J_2 \end{pmatrix}\right) = \mathbb{Z}\left\{\begin{pmatrix} G/H_1 \leftarrow G/K_1 \rightarrow G/J_1 \\ G/H_2 \leftarrow G/K_2 \rightarrow G/J_2 \end{pmatrix}\right\}$$

We have a functor $X : \mathcal{B}\boxtimes\mathcal{B} \to \mathcal{B}$

$$\begin{pmatrix} G/H_1 \\ G/H_2 \end{pmatrix} \longrightarrow G/H_1 \times G/H_2$$

For two Mackey functors $M, N : \mathcal{B} \longrightarrow Ab$

formal

$$M \otimes N : \mathcal{B}\boxtimes\mathcal{B} \to Ab$$

$$\begin{pmatrix} G/H_1 \\ G/H_2 \end{pmatrix} \longmapsto M(G/H_1) \otimes N(G/H_2)$$

We define $M \square N = X_\#(M \otimes N)$

additive

<u>pushforward</u> along the functor $X : \mathcal{B}\boxtimes\mathcal{B} \to \mathcal{B}$.

left Kan extension = left adjoint to pullback X^*.

$\Box$ makes Mackey functors into a tensor category

(coefficients of "ordinary" G-equivariant (co)homology theories)

What is the unit of $\Box$?

Answer: The Burnside Mackey functor:

$\mathcal{A} = \mathcal{A}_G \qquad \mathcal{A}(G/H) = A(H)$

$K\{\cong$ classes of finite H-sets, $\amalg\}$

restrictions in $\mathcal{A}$: passing to a subgroup

corestrictions: $K \times_H$?

Green functor: Commutative (associative unital) ring in the symmetric tensor category of Mackey functors.

Can talk about modules over a Green functor.

In particular, Mackey functors = Modules over $\mathcal{A}$.

Another application of the Burnside category: Constructing enough projectives in G-Mackey functors.

① $\mathcal{A}$ (the Burnside ring Mackey functor) is projective: free on one generator in isotropy G (G/G)

$$\underset{[G/G]}{*} \in A(G)$$

Pushforward functor: $H \hookrightarrow G$

$$\mathrm{Ind}_G^H : \mathcal{B}_H \xrightarrow[G \times_H ?]{} \mathcal{B}_G$$

②

$\mathrm{Ind}_G^H \mathcal{A}_H$ is projective: free on one generator in isotropy H (G/H)

enough projective

Example : $\mathbb{Z}/p$

convention: isotropy $\mathbb{Z}/p$ $(\mathbb{Z}/p/\mathbb{Z}/p)$

isotropy 0 $(\mathbb{Z}/p/\{0\})$

trivial orbit

$$\begin{tikzcd} \mathbb{Z}\oplus\mathbb{Z}[T] \arrow[d, bend right, "T \mapsto p"'] \\ \mathbb{Z} \arrow[u, bend right, "1 \mapsto T"'] \end{tikzcd}$$

$\leftarrow$ free orbit

$\leftarrow$ trivial $\mathbb{Z}/p$-action

$\mathcal{A}$

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "1 \mapsto 1+\gamma+\dots+\gamma^{p-1}"'] \\ \mathbb{Z}[\mathbb{Z}/p] \arrow[u, bend right, "1 \mapsto 1"'] \end{tikzcd}$$

$\leftarrow$ think: free orbit

$\leftarrow$ γ act by γ

$\underline{\mathcal{Y}}_p$

$\mathbb{Z}/p \times \underbrace{\mathcal{A}_{\{0\}}}_{\mathbb{Z}}$

enough projectives in $\mathbb{Z}/p$-Mackey functors

Examples of other $\mathbb{Z}/p$-Mackey functors:

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right] \arrow[r] & \mathbb{Z}\oplus\mathbb{Z}[T] \arrow[d, bend right] \arrow[r] & \mathbb{Z} \arrow[d, bend right, "1"'] \\ 0 \arrow[u, bend right] & \mathbb{Z} \arrow[u, bend right] & \mathbb{Z} \arrow[u, bend right, "p"'] \end{tikzcd}$$

$$0 \to \mathbb{Z}^{\varphi} \to \mathcal{A} \to \underline{\mathbb{Z}} \to 0 \quad \leftarrow \text{constant}$$

coconstant

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "p"'] \arrow[r] & \mathbb{Z}\oplus\mathbb{Z}[T] \arrow[d, bend right] \arrow[r] & \mathbb{Z} \arrow[d, bend right] \\ \mathbb{Z} \arrow[u, bend right, "1"'] & \mathbb{Z} \arrow[u, bend right] & 0 \arrow[u, bend right] \end{tikzcd}$$

$$0 \to \overline{\mathbb{Z}} \to \mathcal{A} \to \mathbb{Z}^{\varphi} \to 0$$

$$\begin{tikzcd} 0 \arrow[r] & \left(0 \atop \mathcal{L}_{p-1} \right) \arrow[r, "1+t+\cdots+t^{p-1}"] & \left(\mathbb{Z} \atop \mathbb{Z}[\mathbb{Z}/p] \right),1 \arrow[r] & p\left(\mathbb{Z} \atop \mathbb{Z} \right),1 \arrow[r] & 0 \end{tikzcd}$$

integral (over $\mathbb{Z}$) reduced regular rep. of $\mathbb{Z}/p$

$$0 \to \tilde{\mathcal{L}}_{p-1} \longrightarrow \underline{\mathcal{L}}_p \longrightarrow \tilde{\mathbb{Z}} \longrightarrow 0$$

$$\begin{tikzcd} 1\left(\mathbb{Z} \atop \mathbb{Z} \right)p \arrow[r, "1+t+\cdots+t^{p-1}"] & \left(\mathbb{Z} \atop \mathbb{Z}[\mathbb{Z}/p] \right) \arrow[r] & \left(0 \atop \mathcal{L}_{p-1} \right) \end{tikzcd}$$

$$0 \to \underline{\mathbb{Z}} \longrightarrow \underline{\mathcal{L}}_p \longrightarrow \tilde{\mathcal{L}}_{p-1} \to 0$$

4-periodicity : Projective resolution of $\underline{\mathbb{Z}}$:

$$\cdots \to \underline{\mathcal{L}}_p \to \underline{\mathcal{L}}_p \to \mathcal{A} \to \mathcal{A} \to \underline{\mathcal{L}}_p \to \underline{\mathcal{L}}_p \to \mathcal{A} \to \mathcal{A}$$

$\tilde{\mathbb{Z}} \quad \mathbb{Z}^q \quad \underline{\mathbb{Z}} \quad \tilde{\mathcal{L}}_{p-1} \quad \tilde{\mathbb{Z}} \quad \mathbb{Z}^q \quad \underline{\mathbb{Z}}$

Exercise: Calculate all Ext between $\underline{\mathbb{Z}}$, $\mathbb{Z}^q$, $\bar{\mathbb{Z}}$, $\tilde{\mathcal{L}}_{p-1}$