Math 696

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Example of calculating ordinary equivariant (co)homology

$G = \mathbb{Z}/p$ β = a faithful 1-dimensional complex representation

$p > 2$ prime.

$$X = S^{n\beta} \qquad ?\ \tilde H_*(S^{n\beta}; \underline{A}_{\mathbb{Z}/p})$$

Burnside Mackey functor

as a spectrum

$$\tilde H^*(S^{n\beta}; \underline{A}_{\mathbb{Z}/p}) = H^*(S^{-n\beta}; \underline{A}_{\mathbb{Z}/p})$$

(RO(G)-graded (co)homology of a point)

β does not matter = different β are interchangeable

as a coeff. system

$$\tilde C_*(S^{n\beta})$$

$$\underbrace{S(n\beta)_+}_{(2n-1)\text{-dim.}} \to S^0 \to S^{n\beta}$$

γ ← generator of $\mathbb{Z}/p$

$$C_*(S(n\beta)) = \mathbb{Z}[\mathbb{Z}/p] \xleftarrow{1-\gamma} \mathbb{Z}[\mathbb{Z}/p] \xleftarrow{1+\gamma+\dots+\gamma^{p-1}} \mathbb{Z}[\mathbb{Z}/p] \cdots \xleftarrow{1-\gamma} \mathbb{Z}[\mathbb{Z}/p]$$

$$\mathbb{Z}[\mathbb{Z}/p] \xrightarrow{\varepsilon} \mathbb{Z}$$

a part of periodic free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}/p]$

$$\tilde C_*(S^{n\beta}):\ \underbrace{\overset{0}{\mathbb{Z}} \xleftarrow{\varepsilon} \mathbb{Z}[\mathbb{Z}/p] \xleftarrow{1-\gamma} \mathbb{Z}[\mathbb{Z}/p] \xleftarrow{1+\gamma+\dots+\gamma^{p-1}} \cdots \xleftarrow{1-\gamma} \mathbb{Z}[\mathbb{Z}/p]}_{\otimes_{\mathbb{Z}/p} \underline{A}}$$

or orbit cat.

To calculate $\tilde H_*(S^{n\beta}; \underline{A})$

value on fixed orbit

where the free generator in isotropy $\mathbb{Z}/p$ goes

$$\underline{A} \xleftarrow{1} \underline{\mathbb{Z}}_p \xleftarrow{1-\gamma} \underline{\mathbb{Z}}_p \xleftarrow{1+\gamma+\dots+\gamma^{p-1}} \cdots \xleftarrow{1-\gamma} \underline{\mathbb{Z}}_p$$

$A(\mathbb{Z}/p) \to \mathbb{Z}\oplus\mathbb{Z} \xleftarrow{T} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{p} \cdots$

$\mathbb{Z} \xleftarrow{1} \mathbb{Z}[\mathbb{Z}/p] \qquad \mathbb{Z}[\mathbb{Z}/p]$

$\xleftarrow{0} \mathbb{Z}$ • isotropy $\mathbb{Z}/p$

• ← isotropy 0: a truncated augmented $\mathbb{Z}[\mathbb{Z}/p]$-res. of $\mathbb{Z}$: $H_* = \mathbb{Z}$ in dim $2n$, 0 below

$\mathbb{Z} \quad 0 \quad \mathbb{Z}/p \quad 0 \quad \dots \quad \mathbb{Z}/p \quad 0 \quad \mathbb{Z}$

"$A(\mathbb{Z}/p)$ ←

$$\tilde H_k(S^{n\beta}; \underline{A}) = \mathbb{Z} \qquad k = 0, 2n$$

'' $A(\mathbb{Z}/p)/T$ $\quad \tilde{H}_k(S^{np}; d) = \mathbb{Z}/p \quad k = 0, 2n$ $\quad 0 < k < 2n$ even $\quad$ 0 below $\quad 2n$

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$\tilde{H}^*(S^{n\beta}; \underline{A})$

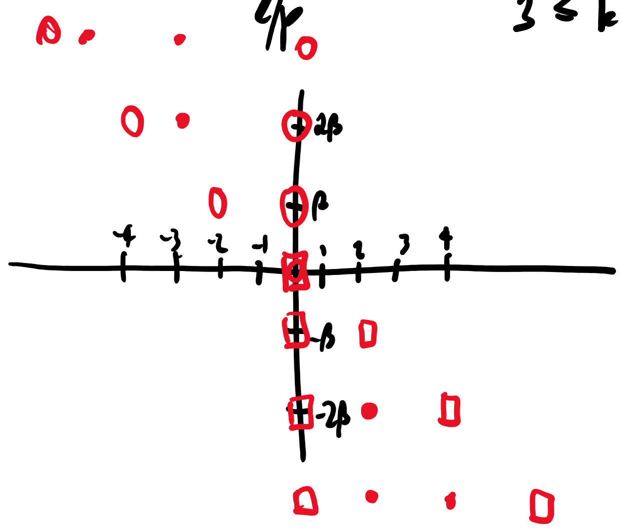
$\mathrm{Hom}_{\mathcal{O}_{\mathbb{Z}/p}}(?, \underline{A})$

$$\mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}[\mathbb{Z}/p] \xleftarrow{1-\gamma} \cdot \xleftarrow{1+\gamma+\dots+\gamma^{p-1}} \cdots\cdots \xleftarrow{1-\gamma} \mathbb{Z}[\mathbb{Z}/p]$$

$$\mathbb{Z}\oplus\mathbb{Z}[T] \xrightarrow{\text{onto}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \cdots\cdots \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$$\mathbb{Z} \quad \underbrace{0 \quad 0}_{\text{gap}} \quad \mathbb{Z}/p \quad \cdots \quad \mathbb{Z}/p \quad \mathbb{Z}$$

$$\tilde{H}^k(S^{n\beta}; \underline{A}) = \begin{cases} \mathbb{Z} & k = 0, 2n \\ \mathbb{Z}/p & 3 \le k < 2n \text{ odd} \\ 0 & \end{cases}$$



$H_\star(*; \underline{A}_{\mathbb{Z}/p})$ ← RO(G)-graded homology

$\boxed{O}$ = $A(\mathbb{Z}/p)$

□ = $\mathbb{Z}$

O = $\mathbb{Z}$ $(p\mathbb{Z})$

• = $\mathbb{Z}/p$

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Equivariant complex cobordism

Let G be a finite group. What is a G-equivariant complex oriented spectrum E?

What is "E-oriented"? "every complex G-bundle is E-oriented"

What is a bundle over a "point"?

$$\overset{\bullet}{G/H} \longrightarrow X$$

G-bundle over this is an H-representation V

Recall: X G-space

$\Pi_1(X)$ = category object: $G/H \longrightarrow X$

morphisms $G/H \times [0,1] \to X$

G/H —— G/K

$$\begin{array}{ccc} G/H & \xrightarrow{\partial_G} & G/K \\ & \searrow \ \ \swarrow & \\ & X & \end{array}$$

Let ξ be an n-dimensional real G-bundle on X

$u \in \tilde{E}^n(X^\xi)$ which restricts to an invertible element over a point $G/H \to X$

it lives in $\tilde{E}^n(S^V)$ ← $\in RO(H)$-graded coefficient of E

For a spectrum E to be complex-oriented, we require that E be a associative commutative unital ring spectrum, and that there exist E-module equivalences $\Sigma^V E \xrightarrow{\simeq} E$ if V is a virtual complex representation of dim. 0.

If $E_H^V \xrightarrow{\sim} E_H$ (as modules) for any virtual cx. H-rep. V of dim. 0, ($H \subseteq G$)
we say E (ACH ring spectrum) is **complex oriented** when every complex G-bundle ξ on X (G-space) of cx. dim. n
is E-oriented.

$u \in \tilde{E}^{2n} X^{\xi}$ restricts to an invertible element in $\tilde{E}^{2n} S^V$ over any $G/H \to X$.

fr fiber over $G/H \to X$, $H \subseteq G$ subgroup

$$\tilde{E}^0_H(S^0) = E^0_H(*)$$

Examples: K_G, MU_G

MU_G: stable G-equivariant complex cobordism

$\neq \Omega_G$ geometrical G-equivariant cx. cobordism