Math 696

Wednesday, April 9, 2025 8:57 AM

Complex-oriented G-equivariant spectrum E: $\leftarrow$ RO(G)-graded (co)homology

- every G-equivariant complex vector bundle is E-orientable.
- For $H \subseteq G$, E_H is $(V - 2\dim_{\mathbb{C}} V)$-periodic for every finite-dimensional complex H-representation V.

needed to formulate

Equivariant formal group laws: Assume that G is finite $\boxed{\text{abelian.}}$

Then it can be shown that E complex orientable $\Leftrightarrow$ the universal G-equivariant complex line bundle is G-orientable.

What space classifies G-equivariant cx. line bundles?

Schubert cells equivariant

$$\mathbb{CP}^\infty_G = \{\text{cx. line through the origin in the complete universe } \mathcal{U}\}$$

$$\mathcal{U} = \bigoplus_{\infty} \bigoplus_{L \in G^*} L$$

$G^* = \{\varphi : G \to \mathbb{C}^\times\}$

L = corresponding rep.

A <u>flag</u> in $\mathcal{U}$: $L_1, L_2, L_3, \dots \in G^*$

each occurs ∞ many times

$\mathcal{U} = L_1 \oplus L_2 \oplus L_3 \oplus \cdots$.

γ_G^1 universal ex. line bundle on $\mathbb{C}P_G^\infty$

$E^* \mathbb{C}P_G^\infty \neq E^*[[x]]$.

$\gamma_G^1 \otimes L$ $\quad x_L \in E^* \mathbb{C}P_G^\infty$ is the Euler class of $\gamma_G^1 \otimes L$

commutative ring R

$\cong E^*\{\{1, x_{L_1}, x_{L_1}x_{L_2}, \dots, x_{L_1}x_{L_2}\cdots x_{L_m}, \dots\}\}$

We put $V_m = L_1 \oplus \cdots \oplus L_m$

$x_{V_m} = x_{L_1} \cdot \cdots \cdot x_{L_m}$

$E^*(*) = A$

$\sum_{m \in \mathbb{N}_0} a_m x_{L_1} \cdots x_{L_m}$

$a_m \in E^*$

G-Equivariant FGL comes from $\mathbb{C}P_G^\infty \times \mathbb{C}P_G^\infty \to \mathbb{C}P_G^\infty$

$\underbrace{\xi, \eta}_{G\text{-equiv. cx. line bundles}} \longmapsto \xi \otimes \eta$

Axioms of G-equivariant FGL: Commutative ring A $= E_{even}(*) = E^{-even}(*)$, commutative A-algebra R $= E^{even}(\mathbb{C}P^\infty_G)$

- regular element $x \in R$, $R/(x) = A$
  non-zero divisor
- $\mathrm{Spf}(R)$ is a commutative (affine) formal group scheme over $\mathrm{Spec}(A)$
  - R has an ideal $I \subset R$ ← ideal of definition, $R = \varprojlim (R/I^n)$ ← complete at I
  - $\psi : R \to R \hat{\otimes}_A R$ ← $E^*(\mathbb{C}P^\infty_G \times \mathbb{C}P^\infty_G)$
    $= \varprojlim (R/I^n \otimes_A R/I^n)$

    commutative associative unital
    (inverse, but follows)

$(\mathbb{C}P^\infty_G)^G = \coprod_{L \in G^*} \mathbb{C}P^\infty$

$$\left[\begin{array}{l} * \xrightarrow{L} \mathbb{C}P^\infty_G \\ A \xleftarrow{i^*_L} R \\ I = \prod_{L \in G^*} x_L \end{array}\right.$$

- A homomorphism of formal group schemes over $\mathrm{Spec}\, A$

  $\lambda : G^*_A \longrightarrow \mathrm{Spf}(R)$

$$\left[\begin{array}{l} \mathcal{O}_{G^*_A} = \mathrm{Map}(G^*, A) \\ \psi : \text{product on } G^* \end{array}\right.$$
← group scheme

product: coordinate-wise

- The ideal of definition $I = \left(\prod_{L \in G^*} x_L\right)$
  where $x_L = \lambda(L) \cdot x$

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Good things that happen: (G finite abelian) can be compact Lie abelian

① E complex oriented G-spectrum $\Rightarrow$ $(E_{even}, E^{even}(\mathbb{CP}^\infty_G))$ is a G-equivariant $\underbrace{\text{formal group law}}_{\text{FGL}}$

② There is an equivariant Lazard ring L_G

$$\mathrm{Hom}_{\mathrm{rings}}(L_G, A) \cong \{\cong \text{ classes of } G\text{-equiv. FGL's } (A, R)\}$$

③ $L_G = (MU_G)_*$!

G-equivariant stable (homotopic) complex cobordism