Math 696

Friday, April 11, 2025 8:58 AM

G-Equivariant complex cobordism (G compact Lie)

Two versions: A complex-oriented G-spectrum MU_G defined via a Thom spectrum. Let $\mathcal{U}$ be complex complete G-universe

$= \bigoplus_{\infty} \bigoplus_{\substack{V \text{ irreducible} \\ \text{complex } G\text{-reps.}}} V$

We create a May G-prespectrum (D_V) $\quad$ V n-dim. cx. G-rep.

$$D_V = \underbrace{\{n\text{-C-vector subspaces in } \mathcal{U} \oplus V\}}_{\text{a model for } BU_G(n)}{}^{\gamma^n}$$

$\leftarrow$ Thom space

over an n-subspace L take the fiber L

If $V \subseteq W$ f.d. (unitary) $\mathbb{C}$-representations of G

orthogonal complement

$$\underbrace{\Sigma^{W-V} D_V}_{\{n\text{-}\mathbb{C}\text{-vector subspaces of } \mathcal{U}\oplus V\}^{\gamma^n \oplus (W-V)}} \longrightarrow D_W$$

$$L \longmapsto L \oplus (W-V)$$

Spectrification $L(D_V) =: MU_G$.

Theorem: MU_G is complex-oriented and for G abelian compact Lie, concentrated in even dimensions, $(MU_G)_* = \pi_* MU_G = L_G$ is the G-equivariant Lazard ring (represent G-equivariant formal group laws).

However, $(MU_G)_n$ is not the n-dimensional G-equivariant complex cobordism group!

this needs Definition

Exercise: G trivial $\Rightarrow$ equivalent to weakly stably complex manifold

G-equivariant smooth (real) compact n-manifolds M without boundary

G acts smoothly

given isomorphism of real G-bundles

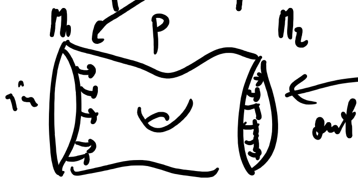
$\tau_M \oplus N \xrightarrow{\cong} \xi$, which is a complex G-equivariant bundle

n-dimensional trivial (G-fixed) N-bundle on $M \Leftarrow M \times \mathbb{R}^N$

trivial G-action on fiber

need this data: G-equivariant unoriented cobordism

Cobordism: $M_1 \sim M_2$ if $\exists$ compact $(n+1)$-dim real G-manifold P, $\partial P = M_1 \sqcup M_2$



$\tau_P|_{M_i} \cong \tau_{M_i} \oplus 1$

$\oplus N \leftarrow$ G-fixed trivial real bundle on P

$\tau_P \oplus N \cong \eta$, restricts to the data on M_i

Friday, April 11, 2025 9:22 AM

G-equivariant complex cobordism groups are the homotopy groups (Z-graded) of a G-spectrum Ω_G

spectrification of (D^g_V) ← geometric

V ← (complex) n-dim. unitary rep. of G

$D^g_V = \{n\text{-vector subspaces } \bigoplus_\infty \mathbb{C} \oplus V\}^{g^m}$ ← Thom space

fiber is $\mathbb{C}$

Computed $(\Omega_{\mathbb{Z}/2})_*$

Theorem (Carlisle), not written down!

(For unoriented G-equiv. cobordism, this was the thesis of A. Wassermann)

unoriented: Conner-Floyd 1960's ... 1960's

ordinary $\mathbb{Z}/2$-equiv. homology

morphism of G-spectra

$\bigoplus_\infty \mathbb{C} \hookrightarrow \mathcal{U}$

$\Omega_G \longrightarrow MU_G$.

What does this map mean geometrically and why are the G-spectra different?

Need to recall non-equivariant Thom's theorem and what fails?

M^{n} Manifold + normal data

S^K sphere

$K \in \mathbb{N}$

$$\Omega \leftarrow U_M^{S^K} \xrightarrow[\text{map}]{\text{classifying}} BU(m) \quad \text{complex}$$

$$S^K \longrightarrow BU(m)^{\gamma^m} \qquad m = K - n$$

Pontryagin-Thom construction

tubular neighborhood $\longmapsto$ total space of γ^m

complement $\longmapsto$ $*$

Works G-equivariantly.

To get back, we use transversality:

$$S^K \longrightarrow BU(m)^{\gamma^m}$$
$$\cup \leftarrow \text{0-section}$$
$$S^K \xrightarrow{f} BU(m)$$

smooth (away from ∞) $\underbrace{\pitchfork}_{\text{transverse}}$ 0-section

$f^{-1}(\text{0-section}) =: M$

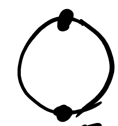
Friday, April 11, 2025 9:36 AM

Transversality can fail equivariantly

Typical example: Let V be a f.d. cx. G-representation.

$$\begin{array}{ccc} * & \xmapsto{G\text{-equiv.}} & * \\ S^0 & \hookrightarrow & S^V \\ 0 & \longmapsto & 0 \end{array}$$

$G = \mathbb{Z}/2$ $V = 2\alpha$ ← complex sign rep.



0 ← only $\mathbb{Z}/2$-fixed point of V

$\mathbb{Z}/2$-equivariantly cannot be moved.

A small addition to the cobordism story:

$MU_n X$, $\widetilde{MU}_n X$

space

based space

data trivialized over $*$

a geometric way to see why cobordism should be a ($\mathbb{Z}$-graded) homology theory

have a geometric interpretation: stably weakly complex n-manifolds M with a map to X

$M \longrightarrow X$

$S^N \longrightarrow BU(m)^{\gamma^m} \wedge X_+$
$\underset{\sim}{\wedge} X$

$$a_V : S^0 \longrightarrow \underset{\uparrow}{S^V} \;\} \in \underbrace{\tilde{\Omega}_{2\dim_{\mathbb{C}} V}(S^V)}_{\pi_{2\dim_{\mathbb{C}} V - V}(\Omega_G)}$$

f.d. complex G-rep.

Theorem: $MU_G = \Omega_G[a_V^{-1} \mid \text{f.d. complex } G\text{-rep. } V]$

(Bröcker–Hook)

$\operatorname{colim}\left(\Omega_G \xrightarrow{a_V} \Sigma^{V - 2\dim_{\mathbb{C}} V}\Omega_G \to \cdots\right)$

use all V's ∞ many times

Next time: will describe $(MU_G)_* = L_G$ for G finite abelian

$\mathbb{Z}/2$-equivariant spectrum

Next topic: Real cobordism $M\mathbb{R} \rightsquigarrow$ Hill, Hopkins, Ravenel solution of the Kervaire problem