Math 696

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Coefficients of stable G-equivariant cobordism $(MU_G)_*$, G finite abelian group

compact Lie abelian $\dots\times S^1\times\dots\times S^1$

$=$ G-equivariant Lazard ring L_G. For a commutative ring A,

$$\mathrm{Hom}_{\text{com. rings}}(L_G, A) \cong \{\cong \text{classes of } G\text{-equivariant FGL } (A,R)\}.$$

<u>Explicitly</u>: the structure of L_G.

Let $\mathcal{U}$ be a complete G-universe, a complete flag:

$L_1, L_2, L_3, \dots$ $\leftarrow$ contains every element of G^* ∞ many times

$\mathcal{U} = L_1 \oplus L_2 \oplus L_3 \oplus \dots$

$V_n = L_1 \oplus \dots \oplus L_n$

$x_{L_n} \in R$ $\qquad x_{V_n} = x_{L_1} \cdots \cdot x_{L_n}$ $\qquad \gamma_G'$

correspond to Thom classes of $(\text{univ. line bundle on } \mathbb{CP}^\infty_G) \otimes L_n$

$R = A\{\{x_{V_0}, x_{V_1}, x_{V_2}, \dots\}\}$ $\leftarrow$ all ∞ sums are allowed

$$R \longrightarrow R/(x) = A$$
$$x_L \longmapsto u_L \quad \text{Euler class}$$

$$R \longrightarrow R \hat{\otimes} R$$
$$x \longmapsto \sum_{i,j\in\mathbb{N}_0} \bar{a}_{ij}\, x_{V_i} \otimes x_{V_j}$$

y_{V_i} $\quad z_{V_j}$

equivariant analog of $F(y,z)$

$y +_F z$

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$\psi : R \to R \hat{\otimes} R$

$x \mapsto \sum_{i,j \in \mathbb{N}_0} \bar{a}_{ij}\, y_{V_i} \otimes z_{V_j}$ "$y +_F z$"

depend on selection of complete flag (V_i)

this becomes a finite polynomial in u_L, $L \in G^*$

What if I substitute u_{L_i} for y or z or both?

writing G^* additively

$L_G = \mathbb{Z}[\bar{a}_{ij}, u_L \mid L \in G^*] / \text{relations}$

What are these relations?

(1) $\underbrace{u_L +_F u_M}_{\text{finite polynomial in } u_{V_i}} = u_{L+M}$

$u_0 = 0$

Recall non-equivariantly: The Lazard ring

$L = \mathbb{Z}[a_{ij}] / \text{relations}$

comm. assoc. unitality of FGL

what if I substitute u_L for y or z (not both?)

I get an infinite "series" containing y_{V_i} with polynomial coeffs. in the u_L's.

$R \longrightarrow R/x = A$

$x_L \longmapsto u_L$

$x \longmapsto 0 = u_{0 \in G^*}$

use this $\left\{ x_L = x +_F u_L = \sum_{n \in \mathbb{N}_0} p_n(u_L \mid L \in G^*)\, x_{V_n} \right.$

$p_0 = 1$

$\bar{a}_{ijk} = \bar{\bar{a}}_{ijk}$

same as non-equiv. relation (replacing $\bar{a}_{ij}$ by a_{ij}) modulo u_L, $L \in G^*$

(2) Ordinary FGL relations $\Rightarrow$ polynomial relations in $\bar{a}_{ij}$, u_L, $L \in G^*$ = non-equivariant relations mod $(u_L \mid L \in G^*)$

$x_{V_i} = x_{L_1} \cdots x_{L_i}$

$x \to \sum_{i,j} \bar{a}_{ij}\, x_{V_i} \otimes x_{V_j} \longmapsto \sum \bar{a}_{ijk}\, x_{V_i} \otimes x_{V_j} \otimes x_{V_k}$

expand this term

$= \sum_{i,j} \bar{a}_{ij}\, x_{V_i} \otimes x_{V_j} \longmapsto \sum \bar{\bar{a}}_{ijk}\, x_{V_i} \otimes x_{V_j} \otimes x_{V_k}$

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Toward Hill-Hopkins-Ravenel solution of Kervaire 1

Real cobordism

First, let us discuss Real K-theory KR.

why is R capital?

First, let it not be: orthogonal K-theory KO. Finite CW-complex X,

$$KO^0(X) = K\{\cong \text{classes of f.d. real vector bundles on } X\}$$

$\|$ ← analogous to the complex case

$$[X, BO \times \mathbb{Z}]$$ ← unbased homotopy classes of maps

What about Bott periodicity? More complicated: $\Omega^8 BO \times \mathbb{Z} \simeq BO \times \mathbb{Z}$

There is an algebraic topology proof, write down maps (Clifford algebras) show $\cong$ in homology H-spaces

Serre S.S.

$$\left\{\begin{aligned}
&\Omega(BO \times \mathbb{Z}) \simeq O \\
&\Omega O \simeq O/U \\
&\Omega O/U \simeq U/Sp \\
&\Omega U/Sp \simeq BSp \times \mathbb{Z} \\
&\Omega(BSp \times \mathbb{Z}) \simeq Sp \\
&\Omega Sp \simeq Sp/U \\
&\Omega Sp/U \simeq U/O \\
&\Omega U/O \simeq BO \times \mathbb{Z}
\end{aligned}\right.$$

Sp_n = compact form of symplectic group; $n \times n$ quaternion matrices "orthonormal" in the sense of quaternions.

Paper by Atiyah: K-theory and Reality

We consider a genuine $\mathbb{Z}/2$-equivariant spectrum KR.

can use it to directly prove 8-periodicity

n	$\pi_n(BO \times \mathbb{Z})$
0	$\mathbb{Z}$
1	$\mathbb{Z}/2$
2	$\mathbb{Z}/2$
3	0
4	$\mathbb{Z}$
5	0
6	0
7	0

(mod 8)

Application of equivariant techniques to a non-equivariant question.

From there: MR, multiplicative norm, MUR