Math 696

Wednesday, April 16, 2025 8:59 AM

Real K-theory KR : $\mathbb{Z}/2$-spectrum

$\boxed{\text{Atiyah: K-theory and reality}}$

X compact $\mathbb{Z}/2$-CW complex

$$KR^0 X := K(\{\cong \text{classes of } \underbrace{\text{Real bundles}}\}, \oplus)$$

univ. ab. group on a commutative monoid

A Real bundle is a complex bundle non-equivariantly with a $\mathbb{Z}/2$-equivariant (real) structure where the generator γ of $\mathbb{Z}/2$ acts by an antiisomorphism:

$$\gamma(\lambda x) = \bar{\lambda}\gamma(x).$$

$\lambda \in \mathbb{C}$

Notice: If $\gamma : \mathbb{C}^n \to \mathbb{C}^n$ is any antiisomorphism $\gamma^2 = 1$ then

$$\gamma(\lambda x) = \bar{\lambda}\gamma(x)$$

the fixed points of γ $\{x \in \mathbb{C}^n \mid \gamma(x) = x\} \cong \mathbb{R}^n$.

(has real dim. n)

(easy exercise: most basic case of Hilbert 90).

If X is a fixed G-space then a Real n-bundle on X is equivalent to a real n-bundle on X.

$$X \ \mathbb{Z}/2\text{-fixed} : \quad KR^0(X) = KO^0(X).$$

Theorem (analogous to non-equivariant case): For a compact $\mathbb{Z}/2$-CW complex X,

$$KR^0(X) \cong [X, \underbrace{BU \times \mathbb{Z}}]$$

$\mathbb{Z}/2$-equiv., unbased homotopy classes $\square$

with $\mathbb{Z}/2$-action by complex conjugation

n-dim. cx. vector subspace $V \subset \bigoplus_{\infty} \mathbb{C}$

apply $x \mapsto \bar{x}$

Same proof

Easy exercise: $(BU \times \mathbb{Z})^{\mathbb{Z}/2} = BO \times \mathbb{Z}$

KR is a good tool for proving $\underbrace{\text{real Bott periodicity}}_{KO \text{ is 8-periodic}}$. $V \longmapsto \bar{V}$

(alt. proofs:
- Harmonic analysis (?)
- alg. topology
- index theory Dirac operator)

Theorem:
Let α be the real sign representation of $\mathbb{Z}/2$. Then

$$\Omega^{\underbrace{1+\alpha}}(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}.$$

$\mathbb{C}$, complex conjug.

Proof:
Same as non-equivariant Atiyah-Bott proof, keeping track of cx. conjugation. $\square$
695

Therefore, $BU \times \mathbb{Z}$ is the 0-space of a genuine $\mathbb{Z}/2$-spectrum KR.

To deduce 8-periodicity:

$$S(k\alpha)_+ \longrightarrow S^0 \longrightarrow S^{k\alpha}$$

mapping cone

$\boxed{\underbrace{k=1,2,4}}$

The "trivial" bundle $k\alpha$ on $S(k\alpha)$ is isomorphic to the trivial bundle k on $S(k\alpha)$

unit sphere in $k\alpha$

$\mathbb{R}^k$ is a division algebra for $k=1,2,4$. Over a point $v \in S(k\alpha)$, multiply by v on the fiber.

For a compact $\mathbb{Z}/2$-CW-cx. X then,

$$\underline{KR^V(S(k\alpha)_+ \wedge X)} \cong KR^{V+k(1-\alpha)}(S(k\alpha)_+ \wedge X), \quad k=1,2,4.$$

$$= KR^{V+2k}(S(k\alpha)_+ \wedge X)$$

(previous Thm - $(1+\alpha)$-periodicity)

$k=1$: $S(\alpha) = \mathbb{Z}/2$

$$KR^*(S(\alpha)_+ \wedge X) = KR^*(\mathbb{Z}/2_+ \wedge X) = K^*(X)$$

2-periodic, nothing new

$$KR^*(S(2\alpha)_+ \wedge X) = KSC^*(X)$$

4-periodic

self-conjug. K-theory

X compact CW-cx., complex bundle with an $\cong$ to its cx. conjugate

not necessarily an involution

S^1 with action by $-$

$$\mathrm{hCoeq}(\mathbb{Z}/2 \overset{1}{\underset{\gamma}{\rightrightarrows}} \mathbb{Z}/2)$$

$k = 4$:

$$S(4\alpha_+) \to S^0 \to S^{4\alpha}$$

$\widetilde{KR}^*(S(4\alpha_+) \wedge X)$ is 8-periodic.

(Fact of linear algebra $S^0 \hookrightarrow S^{3\alpha}$ is null-homotopic in KR-thy).

$\therefore S^0 \hookrightarrow S^{4\alpha}$

$$\begin{array}{ccc} \widetilde{KR}^*(X) & \xrightarrow[=0]{} & \widetilde{KR}^{*-4\alpha}(X) \\ & & \downarrow \\ & & KR^{*-4\alpha}(X) \end{array}$$

Long exact sequence:

$$\xrightarrow{0} \underbrace{\widetilde{KR}^{V-4\alpha+1}(X) \to \widetilde{KR}^{V}(S(4\alpha)_+ \wedge X) \to \widetilde{KR}^V(X)}_{\text{Always a short exact sequence, natural, middle term 8-periodic (already proved)}} \xrightarrow{0} \widetilde{KR}^{V-4\alpha}(X)$$

$\therefore \widetilde{KR}^V(X)$ is 8-periodic

KR: "Appetizer" for Real cobordism $M\mathbb{R}$ and Hill-Hopkins-Ravenel (solution of the Kervaire problem) + Xu

Let us stick to $G = \mathbb{Z}/2$.

$$\begin{array}{ccccc} E\mathbb{Z}/2_+ & \longrightarrow & S^0 & \longrightarrow & \widetilde{E\mathbb{Z}/2} \\ \shortparallel & & & & \shortparallel \\ S(\infty\alpha)_+ & & & & S^{\infty\alpha} \end{array}$$

Let E be a $\mathbb{Z}/2$-spectrum. We can create additional $\mathbb{Z}/2$-spectra

$$\underbrace{E\mathbb{Z}/2_+ \wedge E}_{\substack{E_h \\ \text{Borel homology}}} \qquad \underbrace{F(E\mathbb{Z}/2_+, E)}_{\substack{E_c \\ \text{Borel cohomology}}}$$

$S^0 \leftarrow E\mathbb{Z}/2_+$

$E \to F(E\mathbb{Z}/2_+, E)$

Tate diagram

$$\begin{tikzcd} E\mathbb{Z}/2_+ \wedge E \arrow[r] \arrow[d, "\sim"'] & E \arrow[r] \arrow[d] & \widetilde{E\mathbb{Z}/2} \wedge E \arrow[d] \\ E\mathbb{Z}/2_+ \wedge F(E\mathbb{Z}/2_+, E) \arrow[r] & F(E\mathbb{Z}/2_+, E) \arrow[r] & \underbrace{\widetilde{E\mathbb{Z}/2} \wedge F(E\mathbb{Z}/2_+, E)}_{E_t \text{ - } E\text{-based Tate cohomology}} \end{tikzcd}$$

The Tate diagram:

$$\begin{tikzcd} E_h \arrow[r] \arrow[d, "\mathrm{Id}"'] & E \arrow[r] \arrow[d] & E \wedge S^{\infty\alpha} \arrow[d] \\ E_h \arrow[r, "N"'] & E^h \arrow[r] & E^t \end{tikzcd}$$

norm

The Tate construction in case of E=ordinary homology

computable by spectral sequences.

$S(\infty\alpha)$ has nice increasing filtration by $S(k\alpha)$

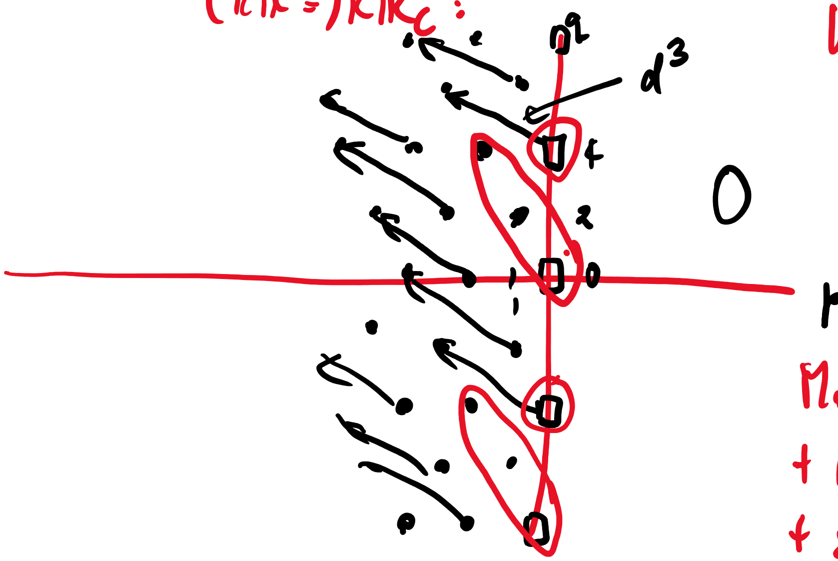
how close is this to an equivalence?

It is an equivalence for $K\mathbb{R}$!

We can write the Borel homology, cohomology and Tate spectral sequences for $K\mathbb{R}$:

write all spectral sequences homologically

$(K\mathbb{R} \simeq) \, K\mathbb{R}_C$:



$\bullet = \mathbb{Z}/2$

$\square = \mathbb{Z}$

More explanation next time
+ Real cobordism
+ slice spectral sequences on $K\mathbb{R}$.