Math 696

Friday, April 18, 2025 8:56 AM

Real K-theory $K\mathbb{R}$ (Atiyah: K-theory and reality)

$\mathbb{Z}/2$-spectrum:

$K\mathbb{R}^0(X) = K(\{\cong$ classes of complex bundles with $\mathbb{Z}/2$-action where the generator acts by antiisomorphism $\bar{\lambda}\gamma(x) = \gamma(\lambda x)\}, \oplus)$

$\mathbb{Z}/2$-compact CW-complex

X fixed: $K\mathbb{R}^0(X) = KO(X)$

$K\mathbb{R}$ is $(1+\alpha)$-periodic (695 proof for K, keep track of antiinvolution)

real sign rep.

last time: 8-periodic

E $\mathbb{Z}/2$-spectrum: $F(E\mathbb{Z}/2_+, E) = E_c$

Borel cohomology spectrum of E

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$E\mathbb{Z}/2 = S(\infty\alpha)$

$\uparrow$ $\mathbb{Z}/2$-CW-complex, cell filtration by $S(n\alpha)$

$\pi_n^{\mathbb{Z}/2} F(\mathbb{Z}/2_+, E) = \pi_n^{\{e\}}(E)$

So we have a spectral sequence:

(equivariant AHSS) $H^p(\mathbb{Z}/2; E^q_{\{e\}}(*)) \Rightarrow \underbrace{E_C}^{p+q}(*)$

coefficient of $E_C = F(E\mathbb{Z}/2_+, E)$.

Apply this to $E = K\mathbb{R}$

$$K\mathbb{R}_n^{\{e\}} = K_n = 0 \quad n \text{ odd}$$
$$= \mathbb{Z} \quad n \text{ even}$$

Grade all spectral sequences <u>homologically</u>

$H^*(\mathbb{Z}/2; \mathbb{Z})$: $\mathbb{Z}$ 0 $\mathbb{Z}/2$ 0 $\mathbb{Z}/2$ 0 $\cdots$

$\uparrow$ triv.

$H^*(\mathbb{Z}/2; \mathbb{Z}^-)$: 0 $\mathbb{Z}/2$ 0 $\mathbb{Z}/2$ 0 $\mathbb{Z}/2$ 0

$\mathbb{Z}/2$ act

$\square$ $\mathbb{Z}$ $-$

0

$\blacksquare$ $\mathbb{Z}$ trivial

0

$\square$ $\mathbb{Z}$ $-$

trivial

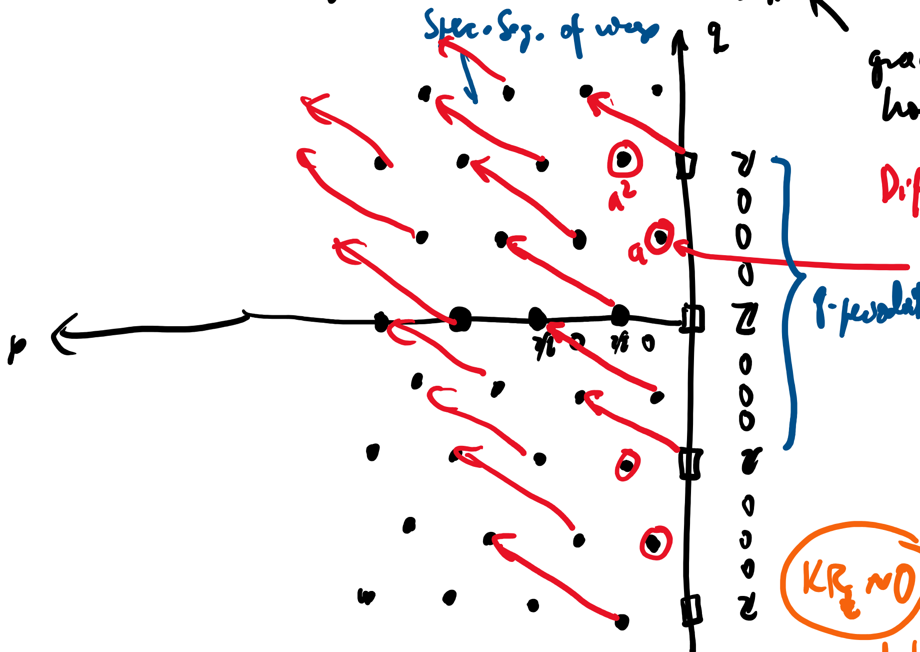
$\square$ $\mathbb{Z}$

$\bullet$ $\mathbb{Z}/2$

AHSS for $F(E\mathbb{Z}/2_+, K\mathbb{R})_*$ ← graded homologically

$\left(K\mathbb{R} \xrightarrow{\sim} F(E\mathbb{Z}/2_+, K\mathbb{R}) \right)$

spoiler → completion theorem



Differentials: d^3

$p=-1,\ q=2$ $\quad p+q=1$

$S^0 \xrightarrow{\varepsilon} S^\alpha \to \Sigma^{-1} K\mathbb{R}$ ← Both periodic

$S^1 \to S^{1+\alpha} \to K\mathbb{R}$

$KR_e \sim O$ $\quad a^3 = 0 \iff \pi_2 O = 0$

$\pi_2 O = \pi_2 O(n) = \pi_2 SO(n) = \pi_2 SO(3) = \pi_2 \underbrace{Spin(3)}_{S^3} = 0$

computable by various S.S. in different regions of the plane

Recall: KO_* : $\mathbb{Z}$ $\mathbb{Z}/2$ $\mathbb{Z}/2$ 0 $\mathbb{Z}$ 0 0 0

$2\mathbb{Z}$
0

Mentioned last time: Tate diag.

$$\begin{array}{ccccc}
E\mathbb{Z}/2_+ \wedge E & \longrightarrow & E & \longrightarrow & S^{\infty\alpha} \wedge E \\
\sim\downarrow & & \downarrow & & \downarrow \\
E\mathbb{Z}/2_+ \wedge F(E\mathbb{Z}/2_+, E) & \to & F(E\mathbb{Z}/2_+, E) & \to & S^{\infty\alpha} \wedge F(E\mathbb{Z}/2_+, E) \\
\| & & \| & & \| \\
E_h & \xrightarrow{N} & E^h & \longrightarrow & E^t
\end{array}$$

Borel homology → E_h $\qquad$ E^t ← Tate cohomology

A variant of the Borel cohomology spectral sequence:

connective cover of a spectrum: cut off negative homotopy groups

$$\tau_{\geq 0} E \longrightarrow E \longrightarrow \tau_{<0} E$$

killing off $\pi_{\geq 0}$ by attaching cells

$\cong$ in π_n, $n \geq 0$ $\cong$ in π_n $n<0$ $\pi_n \tau_{<0} E = 0$ $n \geq 0$

$\pi_n \tau_{\geq 0} E = 0$ for $n < 0$

For $E = K\mathbb{R}$, $\tau_{\geq 0} K\mathbb{R} = k\mathbb{R}$ <u>connective</u> Real K-theory (of course, also non-equivariantly).

$$\Sigma^{1+\sigma} k\mathbb{R} \longrightarrow k\mathbb{R} \longrightarrow H\underline{\mathbb{Z}}$$

Bott (Atiyah) periodicity

recall: $\underline{\mathbb{Z}}$ = constant Mackey functor

$$\underset{\mathbb{Z}}{\overset{\mathbb{Z}}{r=1 \big(\;\; \big) c=2}}$$

This gives another spectral sequence

$$E^1_{**} = H\underline{\mathbb{Z}}_{*+*\alpha} \Rightarrow KR_*$$

$\Sigma^{2(1+\alpha)} H\underline{\mathbb{Z}}$

$\Sigma^{1+\alpha} H\underline{\mathbb{Z}}$

$H\underline{\mathbb{Z}}$

$\Sigma^{-1-\alpha} H\underline{\mathbb{Z}}$

$\Sigma^{-2(1+\alpha)} H\underline{\mathbb{Z}}$

This ends up being smaller than the AHSS. The differential becomes d^1.

For comparison, you can upgrade the S.S. to the

diff. $= d^3$

Hill-Hopkins-Ravenel's slice spectral sequence.

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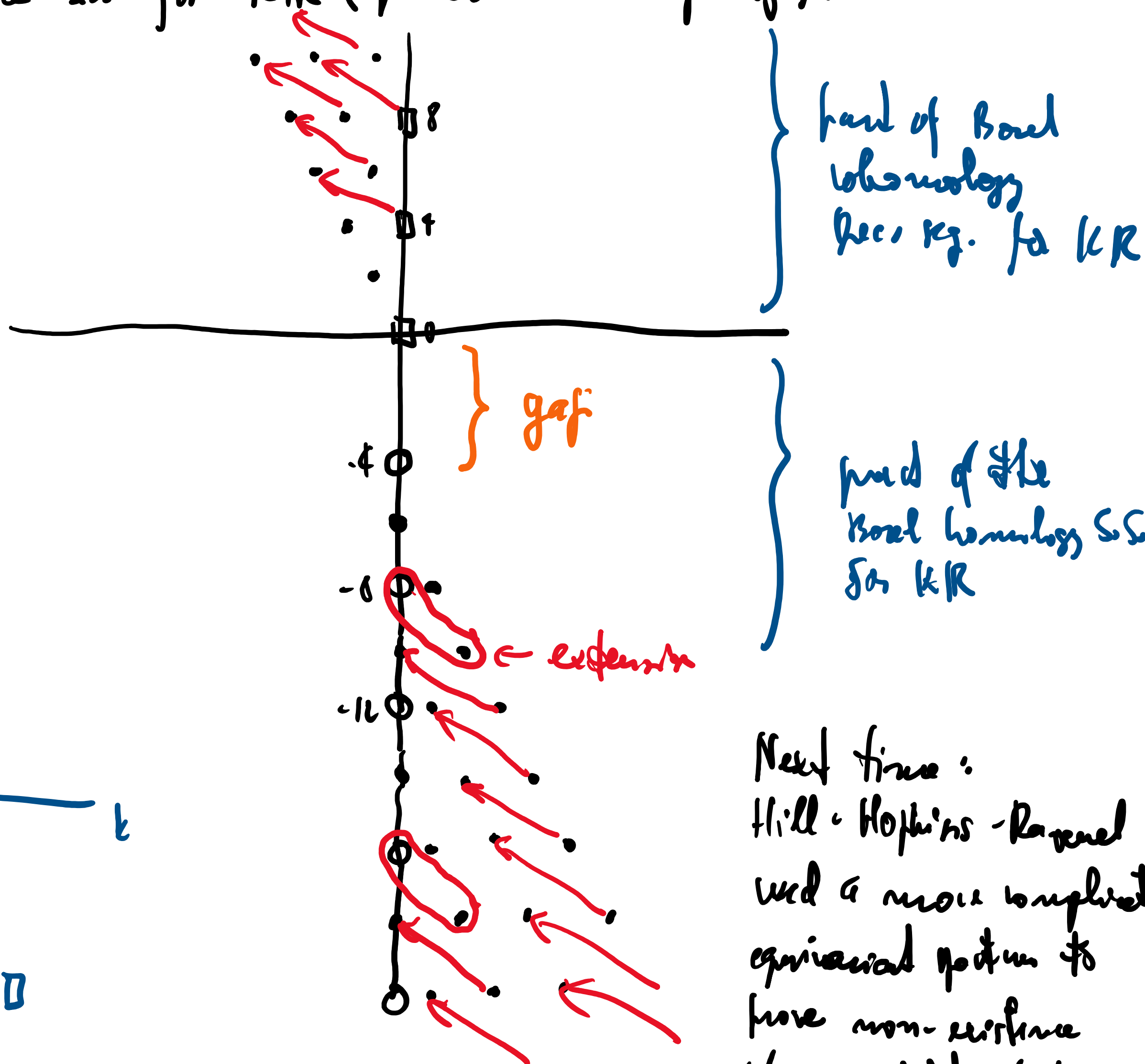
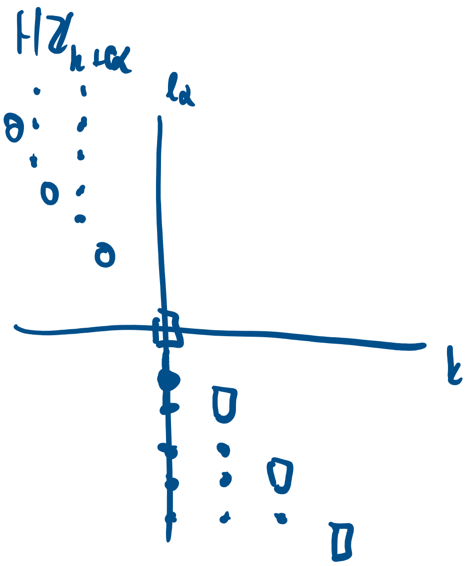
The slice S.S. for $K\mathbb{R}$ (graded homologically):





Next time:
Hill-Hopkins-Ravenel
used a more complicated
equivariant spectrum to
prove non-existence
of manifolds of Kervaire
invariant 1 in dim > 126