Math 696

Monday, April 21, 2025 9:01 AM

The Kervaire invariant 1 problem.

framed manifolds $\rightsquigarrow$ Sphere?

attach handles

$\partial(D^m \times D^{n+1-m})$

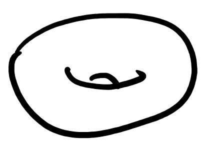
framed cobordism

Obstruction: signature

Kervaire invariant $\in \mathbb{Z}/2$

Browder: It can be only non-trivial in dim $2^k - 2$

Example: in dim. 2



framing: periodic × periodic

by ~~computation~~ intensive computation of $\pi_* S$

? compact symmetric spaces

known to exist: $k = 1, 2, 3, 4, 5, 6, 7$ ← Xu & collaborators

Jones' thesis

unknown

Theorem: Hill, Hopkins, Ravenel: It does not exist for $k \geq 8$.

Real cobordism geometric: $\mathbb{Z}/2$-equiv. manifolds $\subseteq N_1 + N_2\alpha$
Stable normal bundle has the structure of a $\underbrace{\text{Real bundle}}_{\text{= cx. bundle with antilinear } \mathbb{Z}/2\text{-action}}$ real sign rep

from geometry
$RO(\mathbb{Z}/2)$-graded

Same issue as all equivariant cobordisms:

$a : S^0 \underset{c}{\to} S^\alpha$ = manifold of dim. $-\alpha$?!

Stabilise with respect to this (just like in equivariant cobordism):

Thom spectrum $M\mathbb{R} = L(BU_n^{\gamma_n})$ complex conjugation $\Sigma^{\overset{\mathbb{C}}{1+\alpha}} BU_n^{\gamma_n} \to BU_{n+1}^{\gamma_{n+1}}$.

$\mathbb{Z}/2$-structure on MU by complex conjugation

Introduced by Landweber, investigated by Araki (early 80's – introduced a large part of equivariant homotopy thy for $G = \mathbb{Z}/2$).

~1986 LNM 1213 May & al.

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Araki did not calculate $M\mathbb{R}_{k+l\alpha} = M\mathbb{R}_\star$

announced in his last paper that he would do it in the future.

$\underbrace{\text{HHR said it was in unpublished work of Araki.}}_{\text{false}}$

Hu, K.: an elaboration of Atiyah's calculation of $K\mathbb{R}_\star$ via the Borel cohomology spectral sequence. } Completion thm: $\Phi^{\mathbb{Z}/2} M\mathbb{R} = MO$

$M\mathbb{R}_\star$ — Fajstrup

Tate spectral sequence $\rightarrow$ $\widehat{M\mathbb{R}} = MO$

Araki:

Also a theory of FGL. $\rightsquigarrow$ $BP\mathbb{R}$ (at prime 2)

$M\mathbb{R}_{(2)} = BP\mathbb{R}[x_n \mid n \neq 2^h - 1]$

$n(1+\alpha)$.

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element	dim.
v_k	$(2^k-1)(1+\alpha)$
a	$-\alpha$
σ	$(\alpha-1)$

$v_0 = 2$

$$E^1_{**} = \mathbb{Z}/2[v_k, a, \sigma, \sigma^{-1} \mid k \geq 0] \Longrightarrow BPK_\star$$

$v_0 = 2$

minimum term

$\oplus\ \mathbb{Z}/2\, v_R[\sigma^{2^{n+1}}, \sigma^{-2^{n+1}}](a) / a^{2^{n+1}-1}$

differentials = higher analogs of d^3 for KR_*:

$$d_{..}\,\sigma^{2^k} = v_k\, a^{2^{k+1}-1}$$

$$2^k - 2^k\alpha \qquad (2^k-1) + \underbrace{(2^k-1)\alpha - (2^{k+1}-1)\alpha}_{-2^k\alpha}$$

Multiplicative norm (Hu ~2000) $\leftarrow$ motivic homotopy theory

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X H-space

makes sense for a set

$$H\text{-spaces} \xrightarrow{\text{left adjoint}} G\text{-spaces}$$

$$G \times_H X$$

Additive norm

$$H\text{-spaces} \xleftarrow{\text{forget}} G\text{-spaces}$$

$$X^{G/H}$$

Multiplicative norm

twisted

$$H\text{-spaces} \xrightarrow[\text{adjoint}]{\text{right}} G\text{-spaces}$$

$$\prod_{G/H} X$$

Also a version with the smash product

$\rightsquigarrow$ there is a version with May spectra

X H-spectrum on $\mathcal{U}$ ← Complete H-universe

$N_H^G X$ G-spectrum on $\mathbb{C}G \otimes_{\mathbb{C}H} \mathcal{U}$ ← complete G-universe

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$N_{\mathbb{Z}/2}^{\mathbb{Z}/8} M\mathbb{R}$ ← Hill-Hopkins-Ravenel

don't calculate $RO(\mathbb{Z}/8)$-graded coeffs. completely.

dims most relevant: 8 ← real reflection rep. of $\mathbb{Z}/8$ dim. 8

has a lot of rep. ← "only depends on fixed points sizes"

$8-1$ reduced regular rep.

We do actually have $v_{k,8} = N(v_k)$.

Slice spectral sequence also works for $M\mathbb{R}$.

$BP\mathbb{R}$.

also behaves well under N.

can be produced by an equivariant analogue of the Adams tower just like $K\mathbb{R}$.

Ingredients: $H\underline{\mathbb{Z}}$.

constant Mackey functor

HHR: Can detect the Kervaire invariant 1 by

$\pi_* N_{2/2}^{2/8} M\mathbb{R}$

$\pi_* \Delta^{-1} N_{2/2}^{2/8} M\mathbb{R}$

if involves inverting a particular element which makes it 256-periodic (just like inverting v_1 in $k\mathbb{R}$ makes it 8-periodic)

$$\Delta = (N_{2/8}^{2/2} v_4)\cdot(N_{2/8}^{2/4} v_2)\cdot(v_1) \in N_{2/8}^{2/2} M\mathbb{R}_*$$

$M\mathbb{R}_*$ $N_{2/4}^{2/2} M\mathbb{R}_*$

$$|\Delta| = (\underbrace{15 + 3 + 1}_{19})\cdot 8$$

Slice Spectral Sequence $\Rightarrow$ gap in dim $-3,-2,-1$