CHAIN-LEVEL MODELS OF EQUIVARIANT TOPOLOGICAL HOCHSCHILD HOMOLOGY OF SEMIPERFECT AND SMOOTH ALGEBRAS OVER FINITE FIELDS

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ABSTRACT. We find an explicit model for equivariant topological Hochschild homology of semiperfect and smooth algebras over a finite field, as modules over the constant Green functor over a primary cyclic group.

1. INTRODUCTION

In this note, we consider the $\mathbb{Z}/p^r$-equivariant spectrum

$$TR^{r+1}(R) = THH_{\mathbb{Z}/p^r}(R) \tag{1}$$

for an $\mathbb{F}_p$-algebra R. The spectrum $TR^{r+1}(R)$ is automatically a $\mathbb{Z}/p^r$-equivariant E_∞-module over the E_∞-algebra $TR(\mathbb{F}_p) = H\underline{\mathbb{Z}}_p$, the derived category of which is equivalent to the derived category $D\underline{\mathbb{Z}}_p$ of $\mathbb{Z}/p^r$-equivariant Mackey modules over the constant $\mathbb{Z}/p^r$-equivariant Green functor $\underline{\mathbb{Z}}_p$.

The main goal of the present paper is to construct an explicit model of $THH_{\mathbb{Z}/p^r}(R)$ for a smooth (commutative) $\mathbb{F}_p$-algebra R in the (unbounded) derived category of chain complexes of $\underline{\mathbb{Z}}_p$-modules. Our approach is based on the method of faithfully flat descent to quasiregular semiperfect rings. This method was discovered and used by Bhatt, Morrow, and Scholze [3] to construct the motivic filtration on the groups $THH^{\mathbb{Z}/p^r}(R)$ for a smooth $\mathbb{F}_p$-algebra R. By a result of Hesselholt [14], $TR^{r+1}(R)$ in this case is isomorphic to the truncated De Rham - Witt complex of R tensored with a periodicity element of homological degree 2. For quasiregular semiperfect $\mathbb{F}_p$-algebras R, $THH^{\mathbb{Z}/p^r}(R)$ is concentrated in even degrees, so the Postnikov filtration can be used.

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The approach of [3] leads to spectral sequences from crystalline cohomology to several variants of topological cyclic cohomology. A similar situation also occurs in the category of quasisyntomic $\mathbb{Z}_p$-algebras, with crystalline cohomology replaced by prismatic cohomology. Cases where these spectral sequences collapse led to important recent results, including the spectacular progress of Antieau, Krause, and Nikolaus [2] on the algebraic K-theory of $\mathbb{Z}/p^n$. (For other recent results in this direction, see for example A. Mathew [21].)

It should be remarked that from the point of view of homotopy theory, the cases of positive and mixed characteristic are quite different. While, as already remarked, for an $\mathbb{F}_p$-algebra R, (1) is an E_∞-module spectrum over $H\underline{\mathbb{Z}}_p$, which enables a purely chain-level model in the derived category of Mackey modules over $\underline{\mathbb{Z}}_p$, for $\mathbb{Z}_p$-algebras R, this approach only takes us to E_∞-modules over the $\mathbb{Z}/p^r$-equivariant spectrum $TR(\mathbb{Z}_p)$, which is known by Bökstedt and Madsen [5] to be a certain form of connective K-theory (or, perhaps more precisely, J-theory). Accordingly, the Bott periodicity element is clearly visible in the calculations (see e.g. [2, 5, 21]). Thus, modules over connective K-theory would have to be modeled to discuss the mixed characteristic case.

For this reason, we stick to the case of pure characteristic p in the present paper. In this case, on the other hand, our results give a complete information on (1) in terms of classical homological algebra. In principle, then, this refines the information of the spectral sequence [3], which only gives an associated graded object.

Remarkably enough, with the exception of quasiregular semiperfect descent, we are able to use, more or less, the classical method of Hesselholt-Madsen [15] who calculated the answer for a perfect commutative $\mathbb{F}_p$-algebra. We observe that in this case, (1) is given by essentially the "geometric fixed points" of the constant $H\underline{\mathbb{Z}}_p$-module over a primary cyclic group $\mathbb{Z}/p^s$ with respect to the subgroup $\mathbb{Z}/p^{s-r}$ (Theorem 2 below). A key observation is that this does not depend on the choice of $s > r$. Explicit chain-level $\underline{\mathbb{Z}}_p$-Mackey module models are immediately visible. Additionally, geometric fixed points of (1) are also obtained as geometric fixed points of $H\underline{\mathbb{Z}}_p$ with respect to different subgroups.

The other key point is to remark that this result also extends to the case of (1) for quasiregular semiperfect $\mathbb{F}_p$-algebras R, i.e. that for such R, (1) is described as a direct sum of the same building blocks (i.e. geometric fixed points of the $\mathbb{Z}/p^s$-equivariant $H\underline{\mathbb{Z}}_p$ with respect to different subgroups). This can also be proved using the methods of [15], along with the Quillen spectral sequence [22], Theorem 5.1 to start the induction. However, to use the results for the program we set out, it is important to discuss the precise functoriality for the case of quasiregular semiperfect $\mathbb{F}_p$-algebras. For this purpose, we use another device, namely the module-valued Witt vectors defined by Dotto, Krause, Nikolaus, and Patchkoria [6, 7]. Applied to powers of the augmentation ideal J of the limit perfection S_R of a quasiregular semiperfect $\mathbb{F}_p$-algebra R, these are subobjects of $W(S_R)$, which can be used to describe our answer functorially (Theorem 5). Additionally, to show that our functoriality is the right one (thereby describing the correct presheaf on the crystalline site), we use a certain rigidity of the geometric fixed points of $H\underline{\mathbb{Z}}_p$ (Lemma 3).

It is worth noting that other approaches to the motivic filtration are possible, see e.g. Hahn, Raksit, and Wilson [12]. This is related to Hesselholt's description of topological cyclic homology of an associative $\mathbb{F}_p$-algebra as a derived functor [13]. The latter result is based on a description of TR for a unital tensor $\mathbb{F}_p$-algebra as a non-commutative analogue of the De Rham-Witt complex. This is then concentrated in homological degrees $0, 1$. However, it is worth noting that (1) for a unital tensor $\mathbb{F}_p$-algebra, (or of a smooth commutative $\mathbb{F}_p$-algebra), is actually not a direct sum of our building blocks given by geometric fixed points of homology with constant coefficients. Rather, finite $RO(\mathbb{Z}/p^r)$-graded suspensions are involved, which gives non-trivial extensions of the building blocks. This could make discussion of functoriality using this approach substantially more difficult. Therefore, our discussion could be considered as another application of the Bhatt-Morrow-Scholze approach, beyond the motivic filtration.

The present paper is organized as follows: In Section 2, we discuss some basic notation, and recall the theory of Mackey functors over a primary cyclic group, and Mackey modules over the constant Green functor. In Section 3, we introduce our building blocks, and discuss the chain-level model of (1) for a perfect $\mathbb{F}_p$-algebra R. In Section 4, we discuss the chain-level model for quasiregular semiperfect $\mathbb{F}_p$-algebras, and the functoriality.

2. Preliminaries

Equivariant stable homotopy theory for a finite group G is the basic background of our discussion. In particular, we are interested in genuine equivariant spectra, which represent $RO(G)$-graded equivariant homology and cohomology theories. The basic background reference for this topic is Lewis, May, Steinberger [19]. A G-equivariant spectrum is often decorated by a subscript $E = E_G$, while E^G denotes the corresponding fixed point spectrum. For a genuine equivariant spectrum E and a fixed $n \in \mathbb{Z}$, the system of groups $(\pi_n(E^H))_H$ over subgroups $H \subseteq G$ forms an additive functor from the stable orbit category to abelian groups. Such a functor is known as a *Mackey functor*, which can be described purely algebraically (see [8]). Additionally, there exist *equivariant Eilenberg-Mac Lane spectra* HM for a given Mackey functor M, where $(\pi_n(HM^H))_H$ is equal to M for $n = 0$, and equal to 0 else ([20]).

There is a natural tensor product $\square$ in the category of Mackey functors (see [8, 18]). One can further consider (commutative) ring objects in the category of Mackey functors, which are called *Green functors*. One defines Mackey modules over a Green functor in the obvious way. For a Green functor A, on the other hand, HA is an E_∞-ring spectrum, and by the work of Greenlees and Shipley ([11], Section 5), the derived category of E_∞- HA-modules (see [9] for the relevant background) is equivalent to the unbounded derived category of A-Mackey modules.

Recall that for a finite group G and a normal subgroup H, the *geometric H-fixed points* $\Phi^H E$ of a G-spectrum E is the G/H-equivariant spectrum

$$(E \wedge \widetilde{E\mathcal{F}[H]})^H$$

where $\mathcal{F}[H]$ is the family of subgroups of G not containing H, $E\mathcal{F}$ is the classifying space of a family $\mathcal{F}$ (i.e. a G-CW-complex whose K-fixed points are empty if $K \notin \mathcal{F}$ and contractible if $K \in \mathcal{F}$). The symbol $\widetilde{X}$ denotes the unreduced suspension of a G-space X.

From the point of view of representation theory, for a finite-dimensional real G-representation V, one has

$$S^{\infty V} = \widetilde{E\mathcal{F}[V]}$$

where $\mathcal{F}[V]$ is the family of all subgroups $K \subseteq G$ where

$$V^K \neq 0.$$

The essential point is that in the full generality of an A_∞ ring spectrum R, (1) is a genuine $\mathbb{Z}/p^r$-equivariant spectrum. This was first proved by Bökstedt, Hsiang, and Madsen [4]. More modern approaches now exist, see for example [1, 17]. Therefore, it is of interest to us to study the above concepts explicitly for the groups $G = \mathbb{Z}/p^r$.

Specifically, let us recall that a $\mathbb{Z}/p^r$-Mackey functor M can be understood as a sequence of abelian groups

$$M_i,\ i = 0, \dots r,$$

where M_i is an $\frac{\mathbb{Z}/p^r}{\mathbb{Z}/p^i}$-module (the value of the functor M at isotropy $\mathbb{Z}/p^i$). Further, we are given $\mathbb{Z}/p^r$-equivariant abelian group homomorphisms

$$r = r_i : M_i \to M_{i-1},\ c = c_i : M_i \to M_{i+1}$$

(the restriction and corestriction) such that, denoting by γ the generator of the ambient group $\mathbb{Z}/p^r$, we have

$$r_{i+1} \circ c_i = 1 + \gamma^{p^{r-i-1}} + \dots + \gamma^{p^{r-i-1}(p-1)}. \tag{2}$$

Modules over the $\mathbb{Z}/p^r$-equivariant Green functor $\underline{\mathbb{Z}}_p$ are sequences of $\mathbb{Z}_p$-modules M_i, $i = 0, \dots, r$ which satisfy the above axioms and the additional property

$$c_i \circ r_{i+1} = p \tag{3}$$

(see e.g. [18]). Morphisms are tuples of homomorphisms of $\mathbb{Z}_p$-modules which commute with restrictions and corestrictions.

The principal projectives in the abelian category of $\mathbb{Z}/p^r$-equivariant $\underline{\mathbb{Z}}_p$-modules are modules of the form

$$\underline{M}[\mathbb{Z}/p^i] \tag{4}$$

where $0 \leq i \leq r$, M is a projective $\mathbb{Z}_p$-module and

$$\underline{M}[\mathbb{Z}/p^i]_j = \begin{cases} M[\dfrac{\mathbb{Z}/p^r}{\mathbb{Z}/p^j}] & \text{for } r - j \leq i \\ M[\dfrac{\mathbb{Z}/p^r}{\mathbb{Z}/p^{r-i}}] & \text{for } r - j > i. \end{cases}$$

The restrictions and corestrictions are given by

$$r_j(1) = 1 \text{ for } j \leq r - i$$

and

$$c_j(1) = 1 \text{ for } j \geq r - i.$$

(The remaining corestrictions and restrictions are determined.) One can think of (4) as the free $\mathbb{Z}/p^r$-equivariant $\underline{\mathbb{Z}}_p$-module freely generated by M in isotropy $\mathbb{Z}/p^{r-i}$.

3. The building blocks and the case of perfect $\mathbb{F}_p$-algebras

Let α_s be the irreducible complex representation of $\mathbb{Z}/p^s$ given by sending the generator γ to $\zeta_{p^s} = e^{2\pi i/p^s}$. Let $\alpha_{s,i} = \alpha_s^{p^i}$ for $i = 0, \dots, s$. Let, in the derived category $D\underline{\mathbb{Z}}_{\mathbb{Z}/p^r}\text{-}Mod$ of modules over the constant $\mathbb{Z}/p^r$-Mackey functor $\underline{\mathbb{Z}}_{\mathbb{Z}/p^r}$,

$$\mathcal{W}_{r,j} = (\widetilde{C}_*(S^{\infty\alpha_{s,i}}))^{\mathbb{Z}/p^{s-r}},\ i \geq 0,\ i = j - r + s - 1, \tag{5}$$

for $j = 0, \dots, r$. Here $\widetilde{C}_*$ denotes the ordinary reduced (say, singular) chain complex, with G-action induced by the action on the space. This always gives a chain complex of $\underline{\mathbb{Z}}_p$-modules (for a more complete discussion, see [18]).

The definition is ambiguous, but by construction, $\mathcal{W}_{r,j}$ is an E_∞-algebra over $\underline{\mathbb{Z}}_{\mathbb{Z}/p^r}$, whose homotopy type only depends on r and j.

Let us recall that (5) are chain-level models of constructions familiar in equivariant stable homotopy theory. In our present situation, we have

$$(\alpha_{s,i})^{\mathbb{Z}/p^j} = 0 \text{ if and only if } j > i,$$

so

$$S^{\infty\alpha_{s,i}} = \widetilde{E\mathcal{F}[\mathbb{Z}/p^{i+1}]}.$$

Taking fixed points on the level of chain complexes and spectra is equivalent. The reason these complexes show up, and the fact that the definition (5) does not depend on s follows from the cyclotomic property of THH, and from the fact that $TR(\mathbb{F}_p) = H\underline{\mathbb{Z}}_p$.

In more detail, TR is the homotopy inverse limit of $THH^{\mathbb{Z}/p^r}$ with respect to the restriction maps (see [4, 15]). This also automatically becomes a genuine $\mathbb{Z}/p^r$-equivariant spectrum by the isomorphism

$$S^1/(\mathbb{Z}/p^r) \cong S^1.$$

Further, in this sense,

$$TR(\mathbb{F}_p) = H\underline{\mathbb{Z}}_p.$$

(which is proved in [15]). We then obtain a map from the geometric fixed points of $TR_{\mathbb{Z}/p^s}$ to the geometric fixed points of $THH_{\mathbb{Z}/p^s}$, which is an equivalence by an immediate computation. Further, the geometric fixed points of THH is THH again by the cyclotomic property.

Further, by the work of Greenlees and Shipley [11], for a Green functor A, the derived category of HA-modules is equivalent to the unbounded derived functors of A-Mackey modules.

By these considerations, we also have a canonical E_∞-map

$$\pi : \mathcal{W}_{r,j} \to \mathcal{W}_{r,j'},\ 0 \leq j \leq j' \leq r.$$

Comment: One can show that these algebras cannot in general be modeled by strictly (anti-)commutative DGA's. This is because the canonical morphism to the corresponding Tate complex induces an isomorphism on homology groups in non-negative degrees. On the other hand, the Tate complex inherits Steenrod operations from the Borel cohomology complex, which computes the cohomology of a lens space. Therefore, the Tate complex has non-trivial Steenrod operations which restrict to non-trivial Dyer-Lashof operations on the geometric fixed point complex (5).

If we relax our requirements on the coherence of the multiplicative structure, we have the following explicit chain models, where γ denotes a fixed generator of $\mathbb{Z}/p^r$:

$$\mathcal{W}_{r,0} : \underline{\mathbb{Z}} \xleftarrow{p\epsilon} \underline{\mathbb{Z}}[\mathbb{Z}/p^r] \xleftarrow{1-\gamma} \underline{\mathbb{Z}}[\mathbb{Z}/p^r] \xleftarrow{pN} \underline{\mathbb{Z}}[\mathbb{Z}/p^r] \dots \tag{6}$$

where ϵ is the augmentation and

$$N = 1 + \gamma + \cdots + \gamma^{p^r-1},$$

while

$$\mathcal{W}_{r,j} : \underline{\mathbb{Z}} \xleftarrow{\epsilon} \underline{\mathbb{Z}}[\mathbb{Z}/p^{r-j+1}] \xleftarrow{1-\gamma} \underline{\mathbb{Z}}[\mathbb{Z}/p^{r-j+1}] \xleftarrow{N_{r-j+1}} \underline{\mathbb{Z}}[\mathbb{Z}/p^{r-j+1}] \dots \tag{7}$$

where

$$N_i = 1 + \gamma + \cdots + \gamma^{p^i-1}.$$

For any $\mathbb{F}_p$-algebra A, we have an $(\underline{\mathbb{Z}}_p)_{\mathbb{Z}/p^r}$-Mackey module $\underline{W}_{r+1}(A)$ which is $W_{i+1}(A)$ at isotropy $\mathbb{Z}/p^i$ and restrictions resp. corestrictions are given by the Frobenius resp. Verschiebung.

More generally, we also have, for $0 \leq j \leq r$, an $(\underline{\mathbb{Z}}_p)_{\mathbb{Z}/p^r}$-Mackey module $\Phi_j \underline{W}_{r+1-j}(A)$ which is $W_{i+1-j}(A)$ at isotropy $\mathbb{Z}/p^i$ for $i \geq j$ and 0 for lower isotropy, and restrictions resp. corestrictions are given by the Frobenius resp. Verschiebung.

1. **Lemma.** *The $\mathbb{Z}/p^r$-Mackey homology of $\mathcal{W}_{r,j}$ is given by*

$$H_s(\mathcal{W}_{r,j}) = \begin{cases} \Phi_j \underline{W}_{r+1-j}(\mathbb{F}_p) & \text{for } s \geq 0 \text{ even} \\ 0 & \text{else.} \end{cases} \tag{8}$$

Proof. Use (6), (7). □

Comment: Despite the fact that the homology computed in Lemma 1 consists of $\underline{\mathbb{Z}/p^{(r-j+1)}}$-modules, it turns out that $\mathcal{W}_{r,j}$ is not in general equivalent to a chain complex in the category of $\underline{\mathbb{Z}/p^{(r-j+1)}}$-modules.

For example for $r = j = 1$, we have a $\underline{\mathbb{Z}/p}$-module

$$\mathcal{W}_{1,1} \otimes_{\mathbb{Z}} \mathbb{Z}/p. \tag{9}$$

Attempting to construct $\mathcal{W}_{1,1}$ using Bockstein obstruction theory, we get a non-zero obstruction at the $(p-1)$'st step, indicating a non-zero higher Massey power of the generator of an odd-degree cohomology group (one can also see this effect on the Tate complex, where the computation is simpler).

Nevertheless, we have the following:

2. **Theorem.** *Let R be a perfect $\mathbb{F}_p$-algebra. Then in the category of Mackey modules $D\underline{\mathbb{Z}}_p$-Mod, one has*

$$THH_{\mathbb{Z}/p^r}(R) = \mathcal{W}_{r,0} \otimes_{\mathbb{Z}} W(R). \tag{10}$$

Proof. The proof is a complete rehash of the case of $R = \mathbb{F}_p$ treated by Hesselholt and Madsen [15]. We use induction on r using the cyclotomic property, and determining the differential in the Borel homology spectral sequence, which follows from the Tate spectral sequence. This also gives the identification of the connecting map and, by induction the $\underline{\mathbb{Z}}_p$-module model of $THH_{\mathbb{Z}/p^r}(R)$. □

It is also useful to record the morphisms between the objects $\mathcal{W}_{r,j}$ of the (unbounded) derived category of $\mathcal{W}_{r,0}$-modules, which can be thought of as a partial "rigidity" statement:

3. **Lemma.** *The graded module of morphisms from $\mathcal{W}_{r,i}$ to $\mathcal{W}_{r,j}$ in $D\mathcal{W}_{r,0}$ is isomorphic to the $\mathbb{Z}/p^r$ isotropy part of $H_*\mathcal{W}_{r,j}$ when $i \leq j$ and is* 0 *else.*

Proof. Recall the $\mathbb{Z}/p^s$-equivariant based CW-complexes $\Phi_{s,i}$ $0 < i \leq s$ which are equivalent to S^0 in isotropies $\mathbb{Z}/p^j$ for $j \geq i$ and contractible in smaller isotropies. Then the suspension spectra $\Sigma^\infty \Phi_{s,j}$ are E_∞-ring spectra, where

$$\Phi_{s,i} \wedge \Phi_{s,j} \sim \Phi_{s,\max(i,j)}. \tag{11}$$

The groups in question can be expressed as derived $\mathbb{Z}/p^r$-equivariant maps between $\mathcal{W}_{r,0}$-module of the form $\mathcal{W}_{r,0} \wedge \Phi_{s,i}$ for different i. Thus, the positive part of our statement (the case of $i \leq j$) follows from (11).

For the case $i > j$, consider a map of $\mathcal{W}_{r,0}$-modules

$$\Sigma^n \mathcal{W}_{r,i} \to \mathcal{W}_{r,j}. \tag{12}$$

Composing with the n-suspension of the projection

$$\mathcal{W}_{r,j} \to \mathcal{W}_{r,i}, \tag{13}$$

we obtain a map of $\mathcal{W}_{r,0}$-modules

$$\Sigma^n \mathcal{W}_{r,j} \to \mathcal{W}_{r,j}, \tag{14}$$

which we already classified. In particular, we claim that the composition (14) is 0. Otherwise, it would be injective on homotopy groups. However, we also know that the projection (13) is 0 on homotopy groups in degrees $n \gg 0$.

Thus (14) is 0. Thus, the map (12) factors through the cofiber C of the projection (13). However, there are no non-zero morphisms of $H\underline{\mathbb{Z}}$-modules

$$\Sigma^n C \to \mathcal{W}_{r,j}, \tag{15}$$

since C has a $H\underline{\mathbb{Z}}$-module cellular structure where cells are principal projectives with generators in isotropies in which $\mathcal{W}_{r,j}$ has 0 homotopy (see (7)).

However, then there are also no non-zero morphisms (15) of $\mathcal{W}_{r,0}$-modules, since

$$C \wedge_{H\underline{\mathbb{Z}}} \mathcal{W}_{r,0} \sim C.$$

□

4. The case of a semiperfect $\mathbb{F}_p$-algebra

We now treat the case of a quasiregular semiperfect $\mathbb{F}_p$-algebra R. We will give the definition of this concept, which is somewhat technical, following [3], but for our purposes, we can focus on the case

$$R = C \otimes_A C \otimes_A \cdots \otimes_A C \tag{16}$$

where A is a smooth $\mathbb{F}_p$-algebra and C is its colimit perfection.

The reason for this is that if A is a smooth $\mathbb{F}_p$-algebra, we can consider cosimplicial descent from A to (16). A crucial fact used in [3] is that this preserves (1). Therefore, to give a $\underline{\mathbb{Z}}_p$-module chain model of $THH(A)$ for a smooth $\mathbb{F}_p$-algebra A, it suffices to give it for quasiregular semiperfect algebras, specifically those of the form (16).

Denote

$$S_R = \lim(\dots \xrightarrow{\phi} R \xrightarrow{\phi} R)$$

be the limit perfection of R, where $\phi : R \to R$ is the Frobenius.

Denote by J the kernel of the projection $S_R \to R$ and denote also

$$I = Ker(\phi : R \to R).$$

One has

$$J/J^2 \cong I/I^2,$$

since $\phi(I) \subseteq I^2$. Using the notation $L_{B/A}$ of Quillen [22] for the derived cotangent complex for a (non-derived) commutative A-algebra B, we have

$$L_{S_R/\mathbb{F}_p} = 0,$$

which, by Theorem 5.1 of [22], gives

$$L_{R/\mathbb{F}_p} = L_{R/S_R}.$$

Then Theorem 6.3 of [22] gives

$$H_1 L_{R/S_R} = Tor^1_{S_R}(R, R) = J/J^2,$$

which gives

$$H_1 L_{R/\mathbb{F}_p} = I/I^2. \tag{17}$$

By definition ([3], Definition 8.8), a semiperfect $\mathbb{F}_p$-algebra R is called *quasiregular* when I/I^2 is a flat R-module and

$$H_n L_{R/\mathbb{F}_p} = 0 \text{ for } n \neq 1.$$

In our situation of interest, we have a somewhat stronger property. Call a quasiregular semiperfect $\mathbb{F}_p$-algebra R *quasismooth* when J/J^2 is a free R-module of finite rank and

$$Sym^\ell_R(J/J^2) = J^\ell/J^{\ell+1}. \tag{18}$$

4. **Lemma.** *Let A be a smooth $\mathbb{F}_p$-algebra and let P be its colimit perfection. Then*

$$R = P \otimes_A \cdots \otimes_A P$$

is a quasismooth semiperfect $\mathbb{F}_p$-algebra.

Proof. A tensor product over $\mathbb{F}_p$ of two perfect $\mathbb{F}_p$-algebras is perfect (since $\phi \otimes \phi$ is a composition of the injective maps $\phi \otimes 1$ and $1 \otimes \phi$). Then one can prove the analogues of the statements for the ideal J for the kernel of the projection $P \to R$ (since this situation is a colimit of the smooth case, where the statement follows from the theory of parameters). Now while P is different from the limit perfection S_R, one can also prove that the statement about the ideal J is equivalent for the kernel ideal of any surjective homomorphism from a perfect ring. □

Now let R be a quasismooth $\mathbb{F}_p$-algebra, and let J/J^2 be a free R-module on a finite basis B

$$J/J^2 = \bigoplus_B R. \tag{19}$$

We shall write

$$W_r^{B_\ell}(R) = \bigoplus_{B_\ell} W_r(R) \tag{20}$$

where B_ℓ is the set of unordered ℓ-tuples (with possible repeats) of elements of B.

We can write this construction more functorially and more generally as follows. Recalling the notion of Witt vectors with coefficients in a module [6, 7, 23], we have

$$W_r^{B_\ell}(R) \cong W_r(S_R, J^\ell)/W_r(S_R, J^{\ell+1}). \tag{21}$$

We see that the right-hand side does not depend on the choice of a basis, and, in fact, makes sense for every quasiregular semiperfect $\mathbb{F}_p$-algebra R. In that generality, this motivates in fact writing

$$\mathcal{W}_{r,j}(R,\ell) := W(S_R, J^\ell)/W(S_R, J^{\ell+1}) \otimes \mathcal{W}_{r,j}.$$

Thus, we may also write

$$\mathcal{W}_r(R,\ell) := \operatorname{holim} \mathcal{D}_r(R,\ell),$$

where $\mathcal{D}_r(R,\ell)$ is the diagram

$$\begin{array}{ccccccc}
 & & & & & & \mathcal{W}_{r,r}(R,\ell) \\
 & & & & & & \downarrow{\scriptstyle\phi} \\
 & & & & \mathcal{W}_{r,r-1}(R,\ell) & \xrightarrow{\pi} & \mathcal{W}_{r,r}(R,\ell) \\
 & & & & \downarrow{\scriptstyle\phi} & & \\
 & & \dots & \xrightarrow{\pi} & \mathcal{W}_{r,r-1}(R,\ell) & & \\
 & & \downarrow{\scriptstyle\phi} & & & & \\
\mathcal{W}_{r,0}(R,\ell) & \xrightarrow{\pi} & \dots & & & &
\end{array}$$

where ϕ is the map induced by the p-th power (i.e. Frobenius) on R.

5. **Theorem.** *For a quasiregular semiperfect $\mathbb{F}_p$-algera R, one has*

(22)

$$THH^{\mathbb{Z}/p^r}(R)_{2\ell} = \bigoplus_{i=0}^{\ell} W_{r+1}(S_R, J^\ell)/W_{r+1}(S_R, J^{\ell+1}) \otimes_{W_{r+1}(S_R)} W_{r+1}(R),$$

$$THH^{\mathbb{Z}/p^r}(R)_{2\ell+1} = 0.$$

Further, in the category $D\underline{\mathbb{Z}}_p$-Mod, one has

(23) $$THH_{\mathbb{Z}/p^r}(R) = \bigoplus_{\ell\geq 0} \mathcal{W}_r(R,\ell)[2\ell].$$

6. **Corollary.** *If R is a quasiregular semiperfect $\mathbb{F}_p$-algebra, then $THH_{\mathbb{Z}/p^r}(R)$ is equivalent to a direct sum of even suspensions of the $\underline{\mathbb{Z}}_p$-module complexes $\mathcal{W}_{r,i}$.*

Proof of Theorem 5. We first consider the case $r = 0$. Then by [22], Theorem 5.1, applied to the sequence

$$\mathbb{F}_p \to R \otimes R \to R,$$

we obtain

$$L_{R/R\otimes R} = J/J^2[2].$$

By [22], Theorem 6.3, we then have

$$Tor^{R\otimes R}_{2\ell}(R,R) = Sym^\ell_R J/J^2 = J^\ell/J^{\ell+1},$$

$$Tor^{R\otimes R}_{2\ell+1}(R,R) = 0.$$

Now following [15], we have a spectral sequence

$$Tor^{A_*\otimes R\otimes R}_*(R,R) \Rightarrow THH_*(R)$$

where A_* is the dual Steenrod algebra which, using the standard differentials of [15] for $p > 2$, implies (22) (and hence (23)) for $r = 0$.

For $r \geq 1$, one then also repeats the method of [15], considering the Borel homology spectral sequence, which has the differential induced from the case $R = \mathbb{F}_p$ and consequently collapses to even degrees. Extensions are given by multiplication by p composed with the map induced by the Frobenius on R.

One then again gets the total fixed points by using the fundamental cofibration

$$E\mathbb{Z}/p^r_+ \wedge E \to E \to \widetilde{E\mathbb{Z}/p} \wedge E$$

for a $\mathbb{Z}/p^r$-equivariant spectrum E, and identifying the last term by induction using the cyclotomic property. The identification of the connecting map also mimics the perfect case, thus giving the result as well as the decomposition (23).

The induction described is sufficient to prove the statement of Corollary 6, and to count the number of copies of the complexes $\mathcal{W}_{r,i}$. However, we stated our answer functorially in R, and the functoriality must be right in the A_∞-sense to be used for descent. For this purpose, we invoke Lemma 3. We first observe that in the limit, we have

$$TF_{\mathbb{Z}/p^r}(R) \sim \bigoplus_{\ell \geq 0} \underline{W(S_R, J^\ell/J^{\ell+1})}[2\ell]. \tag{24}$$

Since $W(S_R, J^\ell/J^{\ell+1})$ are free $\mathbb{Z}_p$-modules, there are no higher derived maps between the $\underline{\mathbb{Z}}_p$-modules (24). Now let $R \to R'$ be a homomorphism of quasiregular semiperfect $\mathbb{F}_p$-algebras. Then the functoriality we describe fits into the diagram of $H\underline{\mathbb{Z}}_p$-modules

$$\begin{array}{ccc} TF_{\mathbb{Z}/p^r}(R) & \longrightarrow & TF_{\mathbb{Z}/p^r}(R') \\ \downarrow & & \downarrow \\ THH_{\mathbb{Z}/p^r}(R) & \longrightarrow & THH_{\mathbb{Z}/p^r}(R') \end{array} \tag{25}$$

where the horizontal arrows are given by our functoriality, while the vertical arrows are the natural projections. We may extend scalars over $H\underline{\mathbb{Z}}_p = TF_{\mathbb{Z}/p^r}(\mathbb{F}_p)$ to $THH_{\mathbb{Z}/p^r}(\mathbb{F}_p)$ to obtain a diagram of $\mathcal{W}_{r,0}$-modules

$$\begin{array}{ccc} TF_{\mathbb{Z}/p^r}(R) \otimes_{\underline{\mathbb{Z}}_p} \mathcal{W}_{r,0} & \longrightarrow & TF_{\mathbb{Z}/p^r}(R') \otimes_{\underline{\mathbb{Z}}_p} \mathcal{W}_{r,0} \\ \downarrow & & \downarrow \\ THH_{\mathbb{Z}/p^r}(R) & \longrightarrow & THH_{\mathbb{Z}/p^r}(R'). \end{array} \tag{26}$$

By Lemma 3, however, the bottom horizontal arrow (25) is uniquely determined (in the A_∞-sense) by the remaining arrows. This completes the proof of our functoriality statement. □

Comment: Theorem 5 can be thought of as specifying an A_∞-sheaf on the crystalline site of $\mathbb{F}_p$-algebras considered in [3]. Therefore, we can use faithfully flat quasiregular semiperfect descent to obtain Mackey chain models for $THH_{\mathbb{Z}/p^r}(R)$ for a smooth $\mathbb{F}_p$-algebra R.

The "rigidity" of Lemma 3, however, does not imply that in the case of smooth $\mathbb{F}_p$-algebras, we would get again sums of copies of the complexes $\mathcal{W}_{r,i}$. In fact, in the case of the polynomial algebra $\mathbb{F}_p[x]$, it is known ([10, 13, 14, 16]) that one obtains summands of finite $RO(\mathbb{Z}/p^r)$-graded suspensions of the complexes $\mathcal{W}_{r,i}$, which are non-trivial extensions of the complexes $\mathcal{W}_{r,i}$. In the case of quasiregular semiperfect algebras, this behavior is ruled out in part by the fact that the chain homology is contained in even degrees.

However, it is worth noting that by generalizing the Illusie complex to tensor algebras, Hesselholt [13] was able to describe TC of an $\mathbb{F}_p$-algebra as a derived functor. Hahn, Raksit, and Wilson [12] recently used a similar alternative to quasiregular semiperfect descent to define a version of the motivic filtration on commutative ring spectra.

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