PERVERSE MACKEY FUNCTORS

VIKTOR BURGHARDT, PO HU, IGOR KRIZ, AND PETR SOMBERG

Abstract. We discuss, in the language of classical equivariant homotopy theory, the 6-functor formalism in the derived category of G-equivariant spectra for a finite group G, where the 'pure strata' correspond to free spectra over Weyl groups. As a result, we identify new abelian categories of "perverse Mackey functors" associated with arbitrary integral shifts assigned to the individual isotropies. We further prove that their derived categories all coincide with the derived category of Mackey functors. We also compute the abelian categories of perverse Mackey functors for the case of elementary cyclic groups.

1. Introduction

While there have been substantial developments in investigating t-structures on the homotopy category of motivic spectra [?, ?], examples of t-structures on equivariant spectra are less explored. Nevertheless, examples present themselves naturally, e.g. in the case of elementary cyclic groups. For this reason, the authors pursued an analogue of the 6-functor formalism (in the sense of [?, ?, ?]) for isotropy separation in G-equivariant spectra for G a finite group, in relation to the discussion in or near Remark 13.4 in P. Scholze's paper [?]. This allows us to define "perverse" versions of Mackey functors for finite groups G, which we discuss.

There turns out to be a substantial overlap of our formalism with independent previous work by Ayala, Mazel-Gee and Rozenblyum [?, ?] and Cnossen [?]. Nevertheless, it seemed beneficial to describe the idea using the classical language of equivariant stable homotopy theory [?], which is different from [?, ?], and perhaps more familiar to the community. We also describe some of the concrete properties of the "perverse" t-structures obtained, showing for example an analog of Beilinson's theorem for perverse sheaves, which states that the derived

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categories of the hearts of our t-structures are equivalent to the derived category of Mackey functors. (In this paper, all derived categories are unbounded.)

We also characterize explicitly perverse t-structures for elementary cyclic groups, which was our original motivation.

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2. The 6-functor formalism of equivariant spectra and perversity

2.1. **A recollection of equivariant stable homotopy theory.** Let us recall briefly the May setup of G-equivariant spectra for a finite group G [**?**]. One has a derived (otherwise termed "homotopy") category with an underlying topological category, which is the category of based G-spaces with explicit infinite delooping (up to given G-homeomorphisms) with respect to one-point compactifications S^V of finite-dimensional (say, real) representations V of G. Morphisms are systems of maps preserving the structure. This is a topological category with topological limits and colimits and a shift; this gives a notion of homotopy. The functor

$$E \mapsto (E_n)^H$$

(where E is a G-spectrum) has a left adjoint $\Sigma^{\infty-n}G/H_+$, $n \in \mathbb{Z}$. Homotopy classes of maps from these are the homotopy groups $\pi_n^H(E)$, and a weak equivalence is a morphism inducing an isomorphism on those.

The derived category is obtained by inverting the weak equivalences. It can be explicitly built as the full subcategory of homotopy classes of morphisms on cell objects (with cells, again, of the form $\Sigma^{\infty-n}G/H_+$).

By a *family* one means a set of subgroups of G closed under subconjugacy. For a family $\mathcal{F}$, we can consider the classifying space $E\mathcal{F}$ which is a G-CW-complex with a contractible set of H-fixed points for $H \in \mathcal{F}$ and no H-fixed points otherwise. For a family $\mathcal{F}$, one can consider the derived category of $\mathcal{F}$-spectra, which is the category of homotopy classes of morphisms on cell spectra where the cells are of the form $\Sigma^{\infty-n}G/H_+$, $H \in \mathcal{F}$. Such spectra are also known as $\mathcal{F}$-*colocal.*

2.2. **The analogue of open inclusions and proper inclusions.** Now to set up an analog of a 6-functor formalism in this context, an inclusion of families $\mathcal{G} \subseteq \mathcal{F}$ can be considered as an "open inclusion j":

On the derived category of $\mathcal{G}$-spectra, $j^* = j^!$ is defined by

$$j^*E = j^!E = E \wedge E\mathcal{G}_+.$$

The functors $j_!$, j_* are defined by

$$j_!E = E,\ j_*E = F_{\mathcal{F}}(E\mathcal{G}_+, E) = E\mathcal{F}_+ \wedge F(E\mathcal{G}_+, E).$$

(Here $F_{\mathcal{F}}$ denotes the function spectrum in the derived category of $\mathcal{F}$-colocal spectra, i.e. $F_{\mathcal{F}'}(Z, ?)$ is right adjoint to $Z \wedge ?$ in that derived category.)

To give an analog of a "proper inclusion i," denote by $\mathcal{F}[H] = \mathcal{F}[H]_G$ the family of all subgroups of G not containing any conjugate of H. If $\mathcal{F}$ is any family of subgroups of G, we also put $\mathcal{F}_{\mathcal{F}}[H] = \mathcal{F}[H] \cap \mathcal{F}$. We shall also denote by $\mathcal{F}_H$ the family of H-subgroups containing all the elements of $K \in \mathcal{F}$ where $K \subseteq H$, and, when $H \triangleleft G$, we shall denote by $\mathcal{F}/H$ the family of subgroups $K \subseteq G/H$ where $K \cdot H \in \mathcal{F}$.

Now we shall be interested in the following condition:

(1) For $g \in G \smallsetminus N(H)$, no group $K \in \mathcal{F}$ contains both H and $g^{-1}Hg$.

This is always true if H is $\subseteq$-maximal in $\mathcal{F}$, or $H \triangleleft G$. Recall [**?**] that for any subgroup $K \subseteq G$, the forgetful functor from G-spectra to K-spectra has a right adjoint $F_K[G, ?)$ and a left adjoint $G \ltimes_K ?$, which are isomorphic in the derived category by the Wirthmüller isomorphism. Further, for an E_∞-K-ring spectrum E, $F_K[G, E)$ is always a E_∞-G-ring spectrum.

1. **Lemma.** *For a K-E_∞-ring spectrum R and a family $\mathcal{F}$ of subgroups of G, the derived categories of R-module K-spectra (resp. $\mathcal{F}_K$-spectra) and $F_K[G, R)$-module G-spectra (resp. $\mathcal{F}$-spectra) are equivalent.*

Proof. Given an R-module M on K-spectra, we have an $F_K[G, R)$-module spectrum $F_K[G, M)$. Given an $F_K[G, R)$-module N, forget its structure to K-spectra, and then push forward via the counit of adjunction $F_K[G, R)_K \to R$ (which is an E_∞-ring map). One checks that these are inverse equivalences on derived categories. These constructions also induce a correspondence between spectra colocal in the families $\mathcal{F}_K$, $\mathcal{F}$ due to the Wirthmüller isomorphism. □

Now recall that $\widetilde{E\mathcal{F}[H]}$ is an $N(H)$-E_∞-ring spectrum (since it has a model $S^{\infty V}$ for a finite-dimensional $N(H)$-representation V where $V^K \neq 0$ if and only if $K \in \mathcal{F}[H]$). One also has an equivalence of derived categories

$$\begin{array}{c}\mathcal{F}_{N(H)}\text{-}\widetilde{E\mathcal{F}[H]}\text{-module spectra over } N(H)\\ \downarrow \\ (\mathcal{F}_{N(H)})/H\text{-spectra over } W(H)\end{array} \tag{2}$$

given by $X \mapsto X^H$. By Lemma **??**, We also have an equivalence of derived categories

$$\begin{array}{c}\mathcal{F}\text{-}F_{N(H)}[G, \widetilde{E\mathcal{F}[H]})\text{-modules over } G\\ \updownarrow \\ \mathcal{F}_{N(H)}\text{-}\widetilde{E\mathcal{F}[H]}\text{-modules over } N(H).\end{array} \tag{3}$$

We denote by i^* the functor from the derived category of $\mathcal{F}$-spectra over G to the derived category of $(\mathcal{F}_{N(H)})/H$-spectra over $W(H)$ given by the composition of the equivalences of categories (**??**), (**??**) with the pushforward $F_{N(H)}[G, \widetilde{E\mathcal{F}[H]})\wedge ?$. It immediately follows that we have a right adjoint $i_* = i_!$ given by forgetting the $F_{N(H)}[G, \widetilde{E\mathcal{F}[H]})$-module structure, which has a further right adjoint $i^!$ given by

$$F(F_{N(H)}[G, \widetilde{E\mathcal{F}[H]}), ?).$$

By the "open complement" of the "inclusion" of $\mathcal{F}_{N(H)}/H$, we mean simply the inclusion of the family $\mathcal{F}_{\mathcal{F}}[H] \subseteq \mathcal{F}$. Condition (**??**) then gives

$$E\mathcal{F}_+ \wedge \widetilde{E\mathcal{F}[H]} \sim E\mathcal{F}_+ \wedge F_{N(H)}[G, \widetilde{E\mathcal{F}[H]}_{N(H)}),$$

which gives an exact triangle

$$E\mathcal{F}_{\mathcal{F}}[H]_+ \to E\mathcal{F}_+ \to E\mathcal{F}_+ \wedge F_{N(H)}[G, \widetilde{E\mathcal{F}[H]}_{N(H)}). \tag{4}$$

The reader is encouraged, as a warm-up, to consider these constructions for G abelian, here the equivalence (**??**) is not needed.

These 6 functors satisfy the full 6-functor formalism as described in [**?**], pp. 22-23. More precisely, we have the following

1. **Proposition.** *The functor i^* is left adjoint to i_*, the functor j^* is left adjoint to j_*, the functor $i^!$ is right adjoint to $i_!$, the functor $j^!$ is right adjoint to $j_!$. Further, we have*

$$i^* i_*(E) = i^! i_!(E) = j^* j_*(E) = j^! j_!(E) = E \tag{5}$$

where the comparison maps are given by units and counits of adjunction. In the case where

$$\mathcal{G} = \mathcal{F}_{\mathcal{F}}[H],$$

we further have

$$j^* i_*(E) = 0, \tag{6}$$

and units and counits of adjunction induce exact triangles

$$j_! j^!(E) \longrightarrow E \longrightarrow i_* i^*(E) \xrightarrow{[1]} \tag{7}$$

and

$$i_! i^!(E) \longrightarrow E \longrightarrow j_* j^*(E) \xrightarrow{[1]} . \tag{8}$$

Proof. The statements about the adjunctions follow immediately from the definitions. Property (**??**) follows from the fact that

$$E\mathcal{G}_+ \wedge E\mathcal{G}_+ \simeq E\mathcal{G}_+,$$

and the Wirthmüller isomorphism. Property (**??**) must be verified for each case separately. We have

$$j^* j_*(E) = F(E\mathcal{F}_+, E) \wedge E\mathcal{F}_+ = E.$$

On the other hand,

$$j^! j_!(E) = E\mathcal{F}_+ \wedge E$$

which is equivalent to E if E is a cell spectrum with cells whose isotropies are in $\mathcal{F}$.

$$i^* i_*(E) = E$$

follows from

$$\widetilde{E\mathcal{F}[H]}_{N(H)} \wedge \widetilde{E\mathcal{F}[H]}_{N(H)} = \widetilde{E\mathcal{F}[H]}_{N(H)}. \tag{9}$$

The fact that

$$i^! i_!(E) = E$$

follows from (**??**) also and the Wirthmüller isomorphism.

The exact triangles (**??**), (**??**) follow from (**??**), which gives, for a G-spectrum E, an exact triangle

$$E\mathcal{F}_{\mathcal{F}}[H]_+ \wedge E \to E \to F_{N(H)}[G, \widetilde{E\mathcal{F}[H]}_{N(H)})$$

(which is (**??**)) and

$$F(F_{N(H)}[G, \widetilde{E\mathcal{F}[H]}_{N(H)}), E) \to E \to F(\mathcal{F}_{\mathcal{F}}[H]_+, E),$$

which is (**??**). □

2.3. **Gluing t-structures.** When G is a finite group, then, as already remarked, the "pure strata" are free $W(H)$-spectra for subgroups $H \subseteq G$. This is the picture of isotropy separation. So as a result, one can, following the method of [**?**], define "perverse" t-structures on the derived category of G-spectra by shifting the isotropies of different subgroups independently. For a G-spectrum X, and a subgroup $H \subseteq G$, we define

$$j_H^*(X) = (F_{N(H)}[G, \widetilde{E\mathcal{F}[H]}) \wedge X)^H = (\widetilde{E\mathcal{F}[H]} \wedge X_{N(H)})^H,$$

$$j_H^!(X) = F(F_{N(H)}[G, \widetilde{E\mathcal{F}[H]}), X)^H = F(\widetilde{E\mathcal{F}[H]}, X_{N(H)})^H$$

where $E\mathcal{F}[H]$ denotes the classifying space of the family of all subgroups of $N(H)$ not containing H.

More precisely, we have the following result:

2. **Theorem.** *Let G be a finite group and let $\lambda_H \in \mathbb{Z}$ be arbitrary integers. Then there exists a "perverse t-structure" on G-spectra where (graded homologically), the non-negative resp. non-positive categories are the full subcategories on G-spectra X*

$$\mathcal{C}_{\geq 0} = \{X \mid (\forall H \subseteq G)(\forall k < \lambda_H)\ \pi_k^{\{e\}}(j_H^*(X)) = 0\} \tag{10}$$

$$\mathcal{C}_{\leq 0} = \{X \mid (\forall H \subseteq G)(\forall k > \lambda_H)\ \pi_k^{\{e\}}(j_H^!(X)) = 0\}. \tag{11}$$

Proof. For each subgroup H, we have functors j_H^*, $j_H^!$ from G-spectra to free $N(H)/H$-spectra, obtained by taking i^*j^* resp $i^!j^!$ where $j^* = j^!$ is the restriction to a family $\mathcal{F}$ in which H is maximal, and i^*, $i^!$ are the above restriction functors to $\mathcal{F}/H$-spectra, which are the same as free $N(H)/H$-spectra. Using Proposition **??**, the existence of a t-structure satisfying formulas (**??**), (**??**) is essentially proved in [**?**].

To give more detail, one proceeds by induction in the case of one closed stratum. In the present case, this refers to a family $\mathcal{F}$, a maximal group $H \in \mathcal{F}$ (with respect to inclusions), and the family $\mathcal{F}' = \mathcal{F}_H = \mathcal{F} \cap \mathcal{F}[H]$.

In this setting, we assume that we have defined a t-structure on the derived category of G-cell spectra with cells of isotropies in $\mathcal{F}'$, and we also consider the standard t-structure on free $W(H)$-spectra, shifted by λ_H.

We need to prove that a G-cell spectrum E with cells of isotropies in $\mathcal{F}$ factors as

$$\tau_{\geq 0}(E) \to E \to \tau_{\leq -1}(E) \tag{12}$$

where $\tau_{\geq 0}(E) \in \mathcal{C}_{\geq 0}$, $\tau_{\leq -1}(E) \in \mathcal{C}_{\leq -1}$.

To construct the factorization (**??**), we begin with the octahedron

(13)
$$\begin{array}{ccccc}
j_!\tau_{\geq 0}j^!(E) & \longrightarrow & j_!j^!(E) & \longrightarrow & j_!\tau_{\leq -1}j^!(E) \\
{\scriptstyle Id}\downarrow & & \downarrow & & \downarrow \\
j_!\tau_{\geq 0}j^!(E) & \longrightarrow & E & \longrightarrow & E' \\
 & & \downarrow & & \downarrow \\
 & & i_*i^*(E) & \xrightarrow{Id} & i_*i^*(E).
\end{array}$$

We have

$$j^*j_!\tau_{\geq 0}j^!(E) = j^!j_!\tau_{\geq 0}j^!(E) = \tau_{\geq 0}j^!(E)$$

while

$$i^*j_!\tau_{\geq 0}j^!(E) = 0$$

since $i^*j_!$ is the left adjoint to $j^*i_* = 0$. Thus,

$$j_!\tau_{\geq 0}j^!(E) \in \mathcal{C}_{\geq 0}.$$

On the other hand, by the middle horizontal triangle of (**??**),

(14)
$$j^!(E') = \tau_{\leq -1}j^!(E)$$

and hence

(15)
$$j^!E' \in \mathcal{C}_{\leq -1}.$$

We now consider our second octahedron

(16)
$$\begin{array}{ccccc}
i_!\tau_{\geq 0}i^!(E') & \longrightarrow & i_!i^!(E') & \longrightarrow & i_!\tau_{\leq -1}i^!(E') \\
{\scriptstyle Id}\downarrow & & \downarrow & & \downarrow \\
i_!\tau_{\geq 0}i^!(E') & \longrightarrow & E' & \longrightarrow & E'' \\
 & & \downarrow & & \downarrow \\
 & & j_*j^*(E') & \xrightarrow{Id} & j_*j^*(E').
\end{array}$$

Again, we note that

$$i^*i_!\tau_{\geq 0}(i^!(E')) = i^*i_*\tau_{\geq 0}i^!(E') = \tau_{\geq 0}i^!(E')$$

while $j^*i_! = j^*i_* = 0$, so

$$i_!\tau_{\geq 0}(i^!(E')) \in \mathcal{C}_{\geq 0}.$$

On the other hand, just as before, we have

$$i^!(E'') \in \mathcal{C}_{\leq -1},$$

but we also need to verify (**??**) with E' replaced by E''. To this end, however, we evoke the rightmost vertical triangle (**??**). Applying $j^!$, and using the fact that $j^! i_! = j^* i_* = 0$, we get that

$$j^!(E'') = j^! j_* j^*(E') = j^* j_* j^*(E') = j^*(E') = j^!(E'),$$

so we may apply (**??**) directly, and we are done.

□

3. Examples

In this section, we discuss the special case of elementary cyclic groups, and also comparison with known structures, namely suspensions by representation spheres, in the general case.

3.1. **Elementary cyclic groups.** As an example, let us consider the case $G = \mathbb{Z}/p$ for a prime p. Let us begin by writing the condition of a spectrum X being in the heart $\mathcal{C}_0$. Because of shift, without loss of generality, we can assume that $\lambda_{\{e\}} = 0$. Put $\lambda_{\mathbb{Z}/p} = \lambda$. For $H = \{e\}$, we get the condition

$$\pi_k^{\{e\}}(X) = 0 \text{ for } k \neq 0. \tag{17}$$

For $H = G = \mathbb{Z}/p$, we get the conditions

$$\pi_k^{\mathbb{Z}/p}(F(\widetilde{E\mathbb{Z}/p}, X)) = 0 \text{ for } k > \lambda \tag{18}$$

and

$$\pi_k^{\mathbb{Z}/p}(\widetilde{E\mathbb{Z}/p} \wedge X) = 0 \text{ for } k < \lambda \tag{19}$$

We have

$$H\underline{M}[\mathbb{Z}/p] \in \mathcal{C}_0 \tag{20}$$

where for an abelian group M, $\underline{M}[\mathbb{Z}/p]$ is the left Kan extension from the free orbit to the category of Mackey functors (whose value on the free orbit is $M[\mathbb{Z}/p]$ and on the fixed orbit is M). Specifically, (**??**) holds since

$$F(\widetilde{E\mathbb{Z}/p}, \underline{M}[\mathbb{Z}/p]) \sim \widetilde{E\mathbb{Z}/p} \wedge \underline{M}[\mathbb{Z}/p] \sim 0,$$

and thus conditions (**??**), (**??**) are vacuous.

For an abelian group M, we also have a Mackey functor M_ϕ whose value is M on the fixed orbit and 0 on the free orbit. The spectrum HM_ϕ is S^β-periodic for a non-trivial irreducible complex $\mathbb{Z}/p$-representation β, while

$$\widetilde{E\mathcal{F}[\mathbb{Z}/p]} = S^{\infty\beta}.$$

Thus, we conclude that

$$HM_\phi[\lambda] \in \mathcal{C}_0. \tag{21}$$

We also need to consider the constant Mackey functors $\underline{M}$ for an abelian group M (which is M on both the free and fixed orbit, and the restriction is an isomorpism). From the cofibration sequence

$$E\mathbb{Z}/p_+ \to S^0 \to \widetilde{E\mathbb{Z}/p},$$

we conclude that

$$\pi_k^{\mathbb{Z}/p}(F(\widetilde{E\mathbb{Z}/p}, H\underline{M}) = 0 \text{ for } k > -2 \tag{22}$$

and

$$\pi_k^{\mathbb{Z}/p}(\widetilde{E\mathbb{Z}/p} \wedge H\underline{M}) = 0 \text{ for } k < 0. \tag{23}$$

From this, for $p = 2$, if α is the one-dimensional real sign representation of $\mathbb{Z}/2$, we see that

$$H\underline{M}[\lambda\alpha - \lambda] \in \mathcal{C}_0. \tag{24}$$

Since the Mackey functors involved in (**??**), (**??**) and (**??**) generate the category of $\mathbb{Z}/p$-Mackey functors, those equations characterize the t-structure. Hence, we have

2. **Proposition.** *In the case of $G = \mathbb{Z}/2$, the t-structures obtained from our 6-functor formalism are precisely the $(k + \lambda\alpha)$-suspensions of the standard t-structure $k, \lambda \in \mathbb{Z}$.*

In the case of an odd prime p, there is no representation α. Therefore, the t-structure obtained in the case of λ odd is non-trivial. To describe the heart, we will need the following lemma. Let $T \in A(\mathbb{Z}/p)$ be the element of the Burnside ring represented by the free orbit $[\mathbb{Z}/p]$.

3. **Lemma.** *The ring*

$$[\widetilde{\mathbb{Z}/p}, \widetilde{\mathbb{Z}/p}]$$

of stable homotopy self-maps of the unreduced suspension of $\mathbb{Z}/p$, is isomorphic to the ring $\mathbb{Z}[\gamma]/(\gamma^p - 1)$ where γ is the generator of $\mathbb{Z}/p$. Further, in this ring, we have

$$1 + \gamma + \dots + \gamma^{p-1} = p - T. \tag{25}$$

Proof. Consider first the long exact sequence

(26)

$$\dots [S^1, \widetilde{\mathbb{Z}/p}] \to [\Sigma\mathbb{Z}/p_+, \widetilde{\mathbb{Z}/p}] \to [\widetilde{\mathbb{Z}/p}, \widetilde{\mathbb{Z}/p}] \to [S, \widetilde{\mathbb{Z}/p}] \to [\mathbb{Z}/p_+, \widetilde{\mathbb{Z}/p}] \dots$$

The first map is 0. The second $\mathbb{Z}[\mathbb{Z}/p]$-module is

$$\mathbb{Z}[\mathbb{Z}/p]/(1 + \gamma + \dots + \gamma^{p-1}). \tag{27}$$

(We can think of it as being generated by $1-\gamma$.) The last term is 0. The penultimate term is $\mathbb{Z}$, using the cofibration

$$\mathbb{Z}/p_+ \to S \to \widetilde{\mathbb{Z}/p}, \tag{28}$$

with the generator represented by the inclusion $S^0 \subset \widetilde{\mathbb{Z}/p}$. The first statement follows.

For the second statement, one notes that $T^2 = pT$. Thus, we can ask what elements $x \in \mathbb{Z}[\gamma]/(\gamma^p - 1)$ satisfy the equation $x^p = px$. Embedding into $\mathbb{Q}[\gamma]/(\gamma^p-1) = \mathbb{Q}[\zeta_p]\prod\mathbb{Q}$, we expect four solutions, which are

$$p, N, 0, p-N$$

(where $N = 1+\gamma+\cdots+\gamma^{p-1}$). Further, we know from the cofibration sequence (??) that the penultimate map (??) sends T to 0. This narrows down the selection to either 0 or $p-N$. If it were 0, then T would also annihilate the Euler characteristic

$$\chi(\widetilde{\mathbb{Z}/p}) = T-1,$$

which is not the case. This concludes the proof. □

To describe the t-structure from our 6-functor formalism for $\lambda = 1$ for $p > 2$, we remark that in this case, for a Mackey functor Q,

$$Q \in \mathcal{C}_0 \text{ when the corestriction } c \text{ of } Q \text{ is onto} \tag{29}$$

and for an abelian group M,

$$M_\phi[1] \in \mathcal{C}_0. \tag{30}$$

Since for any Mackey functor Q, we can consider the Mackey functor Q' which has the same value as Q on the free orbit, while the value of Q' on the fixed orbit is

$$Im_{c_Q}.$$

We then have a short exact sequence of Mackey functors

$$0 \to Q' \to Q \to M_\phi \to 0 \tag{31}$$

for a suitable abelian group M. Thus, the only elements of $\mathcal{C}_0$ can be expressed as an extension of an object of the form (??) by an object of the form (??).

Accordingly, when $X \in \mathcal{C}_0$, we can consider the connecting map of the long exact sequence associated with the cofibration sequence (??):

$$\partial : \pi_1^{\mathbb{Z}/p}(\widetilde{\mathbb{Z}/p} \wedge X) \to \pi_0^{\{e\}}(X) \tag{32}$$

We find that the corestriction of the part of X of type (??) (which determines it) is the cokernel projection of (??).

On the other hand, the part of X of the form (**??**) can be recovered as $Ker(\partial)$. This implies the following conditions:

$$\partial \text{ is a homomorphism of } \mathbb{Z}[\mathbb{Z}/p]\text{-modules} \tag{33}$$

$$Ker(\partial) \subseteq Ker((1+\gamma+\cdots+\gamma^{p-1})-p) \tag{34}$$

(using Lemma **??**) and

$$Im(\partial) \supseteq Im(1-\gamma). \tag{35}$$

Conversely, one proves that any morphism of $\mathbb{Z}[\mathbb{Z}/p]$-modules satisfying these conditions can be realized by an $X \in \mathcal{C}_0$. Thus, we obtain

3. **Proposition.** *For $\lambda = 1$, the heart $\mathcal{C}_0$ of our perverse t-structure is equivalent to the abelian category of all homomorphism of $\mathbb{Z}[\mathbb{Z}/p]$-modules*

$$\partial : M \to N$$

*which satisfy conditions (**??**) and (**??**).*

□

Note that we can similarly describe the case of λ odd by shifting again by a representation sphere.

3.2. **The general finite group case.**

When G is a general finite group, then we have a parameter $\lambda_H \in \mathbb{Z}$ for every subgroup $H \subseteq G$. The collection $(\lambda_H)_{H\subseteq G}$ is called a *perversity*. As above, the heart of the perverse G-equivariant spectra X with respect to the corresponding t-structure consists of spectra satisfying the conditions

$$\pi_k^H(\widetilde{E\mathcal{F}[H]} \wedge X_{N(H)}) = 0 \text{ for } k < \lambda_H$$

and

$$\pi_k^H(F(\widetilde{E\mathcal{F}[H]}, X_{N(H)})) = 0 \text{ for } k > \lambda_H$$

for all H.

For the standard t-structure, one has $\lambda_H = 0$ for all H. Now let V be a finite-dimensional real H-representations. Then since S^V is invertible, there is a t-structure where $\mathcal{C}_{\geq 0}$ consists of all G-equivariant spectra where

$$\underline{\pi}_k(S^V \wedge X) = 0 \text{ for } k < 0$$

where $\underline{\pi}_k$ denotes the k-th Mackey homotopy group. One readily sees that this corresponds to the above example with

$$\lambda_H^V = -dim_{\mathbb{R}}(V^H). \tag{36}$$

This also makes sense for a virtual real representation V. Such shifts of t-structures on equivariant spectra by suspensions by virtual representations played a role in [?]. Thus, if we denote by $\mathcal{S}_G$ the set of all subgroups $H \subseteq G$, we have a sublattice

$$\Lambda_G \subseteq \mathbb{Z}^{\mathcal{S}_G}$$

consisting of all tuples $\lambda = (\lambda_H)_{H\in\mathcal{S}_G}$ satisfying the condition (??). The set of equivalence classes of perverse t-structures up to suspension by a virtual representation is therefore

$$\mathbb{Z}^{\mathcal{S}_G}/\Lambda. \tag{37}$$

This tends to be non-trivial. Consider, for example, the group $G = \mathbb{Z}/2 \times \mathbb{Z}/2$. All the real representations of G are of real type, and the real irreducible representations are the trivial representation and the three sign representations. Now we notice that for each of these irreducible real representations V, the number

$$\sum_{H\subsetneq G} dim(V^H)$$

is even, equal to 2 for each of the sign representations and equal to 4 for the trivial representations. We conclude that the cardinality of (??) is 2 for $G = \mathbb{Z}/2 \times \mathbb{Z}/2$.

4. An analogue of Beilinson's theorem

4. **Theorem.** *For every perversity $\lambda = (\lambda_H)_{H\subseteq G}$, the derived category of the heart $\mathcal{C}_0$ of the corresponding t-structure is equivalent to the derived category of G-Mackey functors.*

Proof. Denote, for a perversity λ, by $\mathcal{C}_0^\lambda$ the heart of the corresponding t-structure on G-equivariant spectra, and let us also use the symbols $\mathcal{C}_{\geq 0}^\lambda$, $\mathcal{C}_{\leq 0}^\lambda$ accordingly. Then we already remarked that $\mathcal{C}_0^0$ is simply the abelian category of Mackey functors, so the statement is true for $\lambda = 0$.

Also for a virtual representation V, smashing with S^V induces an equivalence between $\mathcal{C}_0^\lambda$ and $\mathcal{C}_0^{\lambda+\lambda^V}$. Since S^V is an invertible object of D-G-Spectra, our statement is hence equivalent for λ and $\lambda + \lambda^V$.

Our general approach to proving the Theorem will follow the approach of Beilinson [?, ?], with the added observation that the material [?] on filtered categories can now be replaced by working in a topological category with homotopically reasonable composition (a treatment using quasicategories can be found in Lurie [?]).

This means doing an induction on families $\mathcal{F}$ on the number of elements of $\mathcal{F}$. To this end, we also need to consider the t-structures on $\mathcal{F}$-colocal spectra constructed above from a perversity λ (depending only on λ_H for $H \in \mathcal{F}$). We will denote the corresponding heart, the subcategory on connective and coconnective objects by $\mathcal{C}_0^{\lambda,\mathcal{F}}$, $\mathcal{C}_{\geq 0}^{\lambda,\mathcal{F}}$, $\mathcal{C}_{\leq 0}^{\lambda,\mathcal{F}}$, respectively.

The case of $\lambda = 0$ deserves special attention. We can refer to objects of $\mathcal{C}_0^{0,\mathcal{F}}$ as $\mathcal{F}$-colocal Mackey functors. They can be modelled by $\mathcal{F}$-colocal spectra X which satisfy

$$\pi_k(X^H) = 0 \text{ for } H \in \mathcal{F},\, k \neq 0. \tag{38}$$

For such a spectrum X, $\underline{\pi}_0 X = M$ is necessarily a Mackey functor, and we may thus choose $X = HM \wedge E\mathcal{F}$. So we see that (**??**) is necessarily the restriction to $\mathcal{F}$ of a Mackey functor. On the other hand, if we denote by $\mathcal{B}$ the Burnside category, then for any additive functor from the full subcategory of $\mathcal{B}$ on objects of $\mathcal{F}$ to abelian groups

$$N : \mathcal{B}|_{\mathcal{F}} \to Ab, \tag{39}$$

the left Kan extension

$$\mathcal{B} \otimes_{\mathcal{B}|_{\mathcal{F}}} N$$

is a Mackey functor whose values on $\mathcal{B}|_{\mathcal{F}}$ coincide with N. Thus, we can identify $\mathcal{C}_0^{0,\mathcal{F}}$ with the category of additive functors (**??**) and additive natural transformations.

We also recall that Mackey functors have enough projectives, given by coproducts of principal projectives $\mathcal{A}_H$, which are free Mackey functos on one element in a given isotropy H. Further, an injective is obtained by tensoring a principal projective with a divisible abelian group, and enough injectives are given by (finite) products of principal injectives. (The reason finite products are sufficient is that there are finitely many isotropies.) The projectives and injectives in Mackey functors also supply enough projectives resp. injectives in $\mathcal{F}$-colocal Mackey functors by applying colocalization to them. As a part of our indcution, we will also construct enough projectives and enough injectives in $\mathcal{C}_0^{\lambda,\mathcal{F}}$. This is a somewhat different situation than in [**?**, **?**, **?**].

In more detail, considering the fact that restrictions of (**??**) to a given isotropy has both a right and left adjoint, monomorphisms resp. epimorphisms are functors (**??**) which are monomorphisms resp. epimorphisms on all isotropies in $\mathcal{F}$. Thus, just as in the case of Mackey functors, free additive functors (**??**) on elements of a given isotropy in $\mathcal{F}$ are principal projectives whose direct sums are enough projectives in $\mathcal{C}_0^{0,\mathcal{F}}$, while tensoring principal projectives with divisible abelian groups are injective, and taking their (finite) products gives enough injectives in $\mathcal{C}_0^{0,\mathcal{F}}$.

Before beginning our induction, first note that, by definition, for $X \in \mathcal{C}^\lambda_{\geq 0}$, $Y \in \mathcal{C}^\mu_{\geq 0}$, we have, by definition,

$$X \wedge Y \in \mathcal{C}^{\lambda+\mu}_{\geq 0}.$$

Smashing the exact triangle

$$\tau^0_{\geq 1} S \to S \to H\mathcal{A}$$

with $X \in \mathcal{C}^\lambda_0$, we therefore get an exact triangle

$$X_1 \to X \to H\mathcal{A} \wedge X, \quad X_1 \in \mathcal{C}^\lambda_{geq1}.$$

Considering the long exact sequence in π^λ_*, this gives

$$\tau^\lambda_{\leq 0}(H\mathcal{A} \wedge X) = X,$$

thus giving a module structure (up to homotopy)

$$H\mathcal{A} \wedge X \to \tau^\lambda_{\leq 0}(H\mathcal{A} \wedge X) \to X.$$

This can be refined to an A_∞-associative odule structure using obstruction theory (by connectivity), which is in turn equivalent to an E_∞-module structure, since $H\mathcal{A}$ is E_∞.

Thus, we obtain an additive functor

$$\mathcal{C}^\lambda_0 \to H\mathcal{A}\text{-Mod}. \tag{40}$$

Using the same method, or an $H\mathcal{A}$-module X, the morphism

$$X \to \tau^\lambda_{\leq 0} X$$

is a morphism of E_∞-$H\mathcal{A}$-modules. Thus, we obtain a lifting of our t-structure to the derived category of $H\mathcal{A}$-modules, which is equivalent to the derived category of Mackey functors. Since this comes from "underlying point-set structures," as mentioned above, we automatically obtain a t-functor

$$F_\lambda : D\mathcal{C}^\lambda_0 \to D\mathcal{A}, \tag{41}$$

where the t-structure on the right-hand side is the one just defined.

By [?], Lemma 1.4, to finish the proof of our theorem, we must show that (??) gives isomorphisms on derived Hom's of objects in $\mathcal{C}_0^{\lambda,\mathcal{F}}$, i.e.

$$Ext^n_{\mathcal{C}_0^{\lambda,\mathcal{F}}}(M,N) \xrightarrow[F_\lambda]{\cong} Ext^n_{\mathcal{C}_0^{0,\mathcal{F}}}(M,N), \qquad M,N \in \mathcal{C}_0^{\lambda,\mathcal{F}}. \tag{42}$$

As advertised, we shall prove (??) for M, N colocal on a family $\mathcal{F}$ by induction on the number of elements of $\mathcal{F}$. Concretely, we shall assume that the statement holds for $\mathcal{F}_{\mathcal{F}}[H]$, and prove it for $\mathcal{F}$. By induction, we have enough projectives and injectives on $\mathcal{F}_{\mathcal{F}}[H]$-colocal objects of $D\mathcal{C}_0^\lambda$. Since $j_!$ preserves projectives and j_* preserves injectives, objects smashed wth $E\mathcal{F}_{\mathcal{F}}[H]_+$ can be factored out from M and N in (**??**).

Thus, to complete the induction step, we need to first of all exhibit the $\mathcal{C}_0^\lambda$-substitute for the $\mathcal{F}$-colocal principal projective $\mathcal{A}_H$. To this end, by the periodicity, we may assume $\lambda_H \leq 0$, and use

$$((\widetilde{E\mathcal{F}_{\mathcal{F}}[H]})_{-\lambda_H} \wedge H\mathcal{A}_H)[\lambda_H] \in \mathcal{C}_0^\lambda \tag{43}$$

where the subscript in the first term means a G-space skeleton (and the tilde denotes unreduced suspensions). Similarly, if $\mathcal{Q}$ is obtained by $\mathcal{A}_H$ by tensoring with a divisible abelian group, we can use

$$F((\widetilde{E\mathcal{F}_{\mathcal{F}}[H]})_{-\lambda_H}, H\mathcal{Q})[-\lambda_H] \in \mathcal{C}_0^\lambda \tag{44}$$

for $\mathcal{C}_0^\lambda$-principal injectives supported in isotropy H.

Regarding resolutions, we see that using (**??**) resp. (**??**) will only differ from $\mathcal{C}_0^0$-resolutions of an object supported in isotropy H by an $\mathcal{F}_{\mathcal{F}}[H]$-colocal object, which will not affect Ext. Thus, the induction is complete. □

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