ON SMITH-STONG'S SELF-CONJUGATE COBORDISM CHALLENGE

PO HU, IGOR KRIZ, BENJAMIN RILEY AND PETR SOMBERG

ABSTRACT. We give a complete algebraic computation of self-conjugate cobordism groups MSC_*, which has been an open problem since the 1960's. Our approach is based on several new ideas, including structured homotopy theory, new aspects of formal group laws and variants of the Adams-Novikov spectral sequence, and the motivic loop of Gheorghe, Isaksen, Wang and Xu. We also obtain new results on cobordism with antilinear involution.

1. INTRODUCTION

On the subject of self-conjugate cobordism, Smith and Stong [54] wrote in 1968: *"To completely discourage any computational desires one has..."* followed by a presentation of the first five groups. Accordingly, in the subsequent decades, a complete computation of self-conjugate cobordism groups was considered hopeless.

The purpose of the present paper is to partially answer this challenge by proving that self-conjugate cobordism groups are isomorphic to the Ext-groups of a certain polynomial ring acting in an explicit way on the complex cobordism ring. In agreement with [54], explicit calculations of these Ext-groups remain very difficult. Our result uses a variety of techniques, including new insights on formal group laws, the philosophy of Gugenheim-May on the cohomology of homogeneous spaces [20], spectral algebra (see e.g. [13, 22, 40, 39]), as well as the motivic loop

---

1991 *Mathematics Subject Classification.* 55P42,55N22,14F42.

Address: Department of Mathematics, Wayne State University, 656 W. Kirby St., Detroit, MI 48202, USA (Hu),

Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109-1043, USA (Kriz and Riley),

Mathematical Institute of Charles University, Sokolovska 83, Prague, Czech Republic (Somberg),

email: kriz.igor@gmail.com (Kriz)

The authors acknowledge support by the grant GA CR 19-28628X. Benjamin Riley was supported by NSF RTG grant DMS 1840234, Hu was supported by NSF grant DMS-2301520, Kriz also acknowledges the support of Simons Foundation award 958219.

technique of Gheorghe, Isaksen, Wang, and Xu [14, 15, 30] (for recent related developments, see also [2, 10, 16, 49, 11]).

To present the history in a more ordered fashion, the problem of computing the ring of complex cobordism groups MU_*, i.e. the complex version of Thom's problem [55], was solved in 1960 by Milnor [41] and Novikov [45, 47]. The *complex cobordism spectrum* MU, along with Atiyah's *K-theory* [4], were two generalized cohomology theories which played a key role in the development of modern algebraic topology.

Studying the effect of *complex conjugation* on these theories was a next logical step. This can be done in at least two different ways: It is possible to consider *self-conjugate structures*, which are complex structures (e.g. manifolds or bundles) equipped with an antilinear automorphism. On the other hand, one can make an antilinear involution a part of the structure and study it $\mathbb{Z}/2$-*equivariantly.* This is often referred to as a *Real* structure.

In the case of K-theory, the two approaches are related in a relatively simple way. Real K-theory $K\mathbb{R}$ was introduced by Atiyah in his elegant proof of real Bott periodicity [5]. Self-conjugate K-theory was considered and calculated by Anderson [1] and Green [19]. Speaking equivariantly, self-conjugate K-theory KSC turns out to be the cofiber of the Euler class of the complex sign representation on $K\mathbb{R}$, thus explaining it completely.

In cobordism, the situation is rather different. The $\mathbb{Z}/2$-equivariant *Real cobordism spectrum* $M\mathbb{R}$ was defined by Landweber [34], and further investigated by Araki [3]. The coefficient ring $M\mathbb{R}_\star$ was eventually calculated by Hu and Kriz [25], and later used, along with its variants, by Hill, Hopkins and Ravenel [21] in their solution of the Kervaire-Milnor problem. It is worth noting that due to the behavior of equivariant transversality, the equivariant Thom spectrum $M\mathbb{R}$ does not actually calculate the corresponding cobordism groups of manifolds, which were characterized by Hu [24] as homotopy groups of suspension spectra, making them essentially uncomputable by current methods.

The spectrum MSC of *self-conjugate cobordism*, on the other hand, was studied by Smith and Stong [54]. It is defined as the Thom spectrum of the pullback of the tautological element of K^0BGL to BSC, which is defined as the homotopy equalizer

$$BSC \longrightarrow BGL \underset{Id}{\overset{A\mapsto (A^T)^{-1}}{\rightrightarrows}} BGL \tag{1}$$

(BGL is homotopically equivalent to BU, but we write BGL to emphasize the fact that the construction makes sense algebraically, which will be relevant below.)

Despite the analogy with KSC, the spectrum MSC is *not* a part of the $\mathbb{Z}/2$-equivariant Real cobordism spectrum $M\mathbb{R}$. In fact, MSC turns out to be much more complicated than $M\mathbb{R}$. For example, the group MSC_1 is $\mathbb{Z}/4$ [18]. To give another example, the image of the map $MSC_4 \to MU_4$ is generated by an Enriques surface (by computation of Chern numbers). The homotopy groups of the spectrum MSC, which are the geometrically defined cobordism groups of manifolds with self-conjugate complex structures on their normal bundles, resisted calculation until the present time. It is worth noting that these groups have no p-torsion for $p \neq 2$, and their localization at $p \neq 2$ was in fact calculated by Smith and Stong [54]. The challenge lies in calculating their 2-primary torsion.

The goal of this paper is to introduce a completely algebraic machine (using the homological algebra of formal groups) which gives a complete calculation of MSC_*, even though the answer is too complicated to be written out in closed form.

Concretely, using formal group law theory, we construct an explicit action of a certain polynomial algebra $A = \mathbb{Z}[\alpha_1, \alpha_2, \alpha_3, \dots]$ on MU_* where α_i has topological degree $-2i$ (graded homologically), and a spectral sequence

$$E_2^{s,t} = Ext_A^{s,t}(\mathbb{Z}, MU_*) \Rightarrow MSC_*. \tag{2}$$

Our main result is

1. **Theorem.** *The spectral sequence (2) collapses and there are no $\mathbb{Z}$-multiplicative extensions. In other words, the self-conjugate cobordism groups are given by*

$$MSC_n = \bigoplus_{t-s=n} Ext_A^{s,t}(\mathbb{Z}, MU_*). \tag{3}$$

The spectral sequence (2) comes from a certain descent resolution of MSC by copies of MU, which we construct using spectral algebra [13, 22, 40, 39]. The key point is that our spectral sequence is *different* from the classical Adams-Novikov spectral sequence [46]. This possibility is enabled by the fact that MU_*MSC is not a flat MU_*-modules. Our spectral sequence can be considered as being based on a "fully derived" MU-resolution. However, having an explicit model for the resolution gives us an explicit flat Hopf algebroid whose *Cotor* is the E_2-term, and whose structure maps we can understand. After some additional manipulations, we see that, at least locally at 2, the E_2-term has the extremely formally simple form shown in (2). More detail of this outline is given in section 2 below.

An important feature of the spectral sequence (2) is that the polynomial algebra A acts trivially on the unit of the ring MU_*, and thereby the polynomial generators of A give rise to Ext^1-elements. These elements are, in fact, permanent cycles (represented by real projective spaces and Landweber manifolds). This form of the spectral sequence (2) is, in fact, formally similar to the situation of Gugenheim-May [20], which contains a theorem computing the cohomology of certain homogeneous spaces by collapse without extensions of a spectral sequence, similarly as in Theorem 1. A substantial difference is that [20] deals with ordinary cohomology, and the collapse theorem of that paper uses differential graded algebra. In our present context, we are dealing with spectral algebra, which is quite a bit more complicated.

A key new tool we use is the fact that our situation has an analogue in motivic homotopy theory over $\mathbb{C}$ [42, 27, 26], for which a link with differential graded algebra was discovered by Isaksen [29] (see also Gheorghe [14], Gheorghe, Wang and Xu [15]). This method was used by Isaksen, Wang, and Xu [30], as well as Burkland and Xu and others (see e.g. [11]) to make new computations of homotopy groups of spheres. Other recent developments and applications of these techniques are described for example in [2, 6, 10, 16, 49]. An important ingredient of these applications is the use of comparisons involving the classical Adams spectral sequence. In the present paper, we use this method in a somewhat new way, exploiting directly a link between diffferentials in our spectral sequence and the appropriate concept of torsion in the motivic situation.

The present paper is organized as follows: In Section 2, we outline the philosophy of our proof, highlighting the main new ideas. In Section 3, we describe the Hopf algebroid in a simpler case of the spectrum $MO[2]$, which is a precursor to our main computation. In Section 4, we discuss the case of the Hopf algebroid for MSC. In Section 5, we prove Theorem 1. Finally, in the Appendix, we recall some classical results which have been obtained by previous authors, and which are implicit in our discussion.

2. The main ingredients of the proof of Theorem 1

Our proof of Theorem 1 is based on several modern techniques which were not available at the time when the problem of computing MSC was first proposed [54]. Because of this, we use the present section to outline the new ideas first before presenting our argument in detail.

2.1. **Spectral commutative rings.** In the construction of our spectral sequence, we use *strictly commutative spectral rings* (i.e. E_∞-ring spectra [38, 13]) In that category, we have a morphism

$$MSC \to MU,$$

and our spectral sequence is obtained by taking the cobar resolution of MU over

$$MU \wedge_{MSC} MU.$$

The E_1-term is the cobar complex of a *Hopf algebroid* (i.e. coordinate ring of an affine groupoid scheme) which is flat and has a surprisingly simple description. At least 2-locally, (which is the only non-trivial prime in our case), its *Cotor* (i.e. E_2-term of our spectral sequence) has the form (2).

In fact, to make these conclusions, we also study the spectrum $MO[2]$, which represents the cobordism of manifolds M with a real bundle ν and an isomorphism

$$\tau_M \oplus \nu \oplus \nu \cong N$$

where N is a trivial bundle. Then $MO[2]$ can be described as the spectrum associated with the prespectrum (D_{2n}) where

$$D_{2n} = BO(n)^{2\gamma^n_{\mathbb{R}}} \tag{4}$$

and the connecting map $\Sigma^2 D_{2n} \to D_{2n+2}$ given by the obvious isomorphism of the pullback of $2\gamma^{n+1}_{\mathbb{R}}$ to $BO(n)$ with $2\gamma^n_{\mathbb{R}} \oplus 2$. We refer to $MO[2]$ as the *double real cobordism spectrum*. This spectrum was studied before, although not extensively (see for example Kitchloo and Wilson [32]).

To connect with MSC, note that one can think of $BO(n)$ as the classifying space of complex n-bundles with an antilinear involution, so (4) can also be characterized as $BO(n)^{\gamma^n_{\mathbb{C}}}$ where $\gamma^n_{\mathbb{C}}$ is the universal complex n-bundle. On the other hand, $BSC(n)$ is the classifying space of complex n-bundle with an antilinear automorphism which is not necessarily an involution. Thus, MSC is defined in the same way, with (4) replaced by $BSC(n)^{\gamma^n_{\mathbb{C}}}$.

Complexification $BO(n) \to BU(n)$ therefore induces a canonical morphisms of E_∞-ring spectra

$$MO[2] \to MSC \to MU. \tag{5}$$

We first calculate the simpler Hopf algebroid $(MU_*, MU \wedge_{MO[2]} MU_*)$, which is a precursor for the spectral sequence (2).

2.2. **Rectified Adams-Novikov spectral sequences.** The spectral sequences for calculating MSC_* and $MO[2]_*$ are similar to the Adams-Novikov spectral sequence in the sense that they use a resolution by free MU-modules, but the cobar complex is not the same as the Adams-Novikov cobar complex. This behavior arises due to the fact that MU_*MSC, $MU_*MO[2]$ are not flat MU_*-modules (see the Comment under Theorem 6 below).

We call our spectral sequence *rectified Adams-Novikov spectral sequences* because their E_2-terms are described as Ext-groups of particularly nice new Hopf algebroids

$$(MU_*, (MU \wedge_{MO[2]} MU)_*) = (L, LS) \tag{6}$$

and

$$(MU_*, (MU \wedge_{MSC} MU)_*) = (L, LSC). \tag{7}$$

It is, essentially, a derived form of the Adams-Novikov spectral sequence. However, finding a purely algebraic description of those Hopf algebroids is a key part of our method. The description is based on carefully studying the formal group law structures involved. The idea of using additional structure on formal group laws in the investigation of MSC is not new (see, notably, Buchstaber and Novikov [8, 9]). However, the structure of 2-*valued formal group laws* used in [8, 9] is not a perfect match here. What is needed for investigating MSC is in fact precisely the structure represented by our novel Hopf algebroids. These structures are new and quite subtle (see Comment at the end of Section 3). Calculating (6) leads to a calculation of (7) using the Witt construction and structural results on bipolynomial Hopf algebras [51].

In the case of MSC, the particularly simple E_2-term then allows an argument proving collapse. The situation is similar to the paper by Gugenheim and May [20], although the present situation is homotopical, not homological, and therefore methods of spectral algebra, which are quite different from the methods of differential graded algebra used in [20], are required.

While the calculation of the Hopf algebroid (6) is a crucial step in calculating the Hopf algebroid (7), the E_2-term of the resulting spectral sequence for $MO[2]$ is more complicated than for MSC, and we do not have a collapse theorem in that case. It is worth noting that we do not really know any differentials in the case of $MO[2]$, and certain surprising elements survive (see Comment at the end of Section 5). Therefore, the possibility of collapse of the rectified Adams-Novikov spectral sequence for $MO[2]$ remains an interesting question.

2.3. **The motivic loop and Novikov formality.** The last ingredient in the proof of Theorem 1 is the recent *motivic loop* technique of Gheorghe, Isaksen, Wang and Xu [14, 15, 29, 30]. Isaksen [29] noted that the homotopy groups of the $\mathbb{C}$-based motivic spectrum S^{Mot}/τ are isomorphic to the E_2-term of the classical Adams-Novikov spectral sequence. Gheorghe [14] proved that S^{Mot}/τ is an E_∞ ring spectrum, and showed that the isomorphism preserves higher products. Gheorghe, Wang, and Xu [15] established a categorical level equivalence.

Theorem 1 identifies MSC as another example (in addition to the trivial example MU) of an ordinary spectrum with a similar behavior, i.e. for which a derived Adams-Novikov resolution gives a complete calculation of homotopy groups. This suggests a term *Adams-Novikov-formal spectra* for such examples, even though it is at present difficult to give a precise definition.

While the theory of Gheorghe, Isaksen, Wang, and Xu involves *motivic spectra*, it has made a large impact on computations of classical stable homotopy groups. Isaksen, Wang and Xu [30] (see also Burkland, Xu [11]) used this machinery to greatly expand the known calculations of stable homotopy groups of spheres. The case of MSC is somewhat different: it is simpler in the sense that the algebraic objects involved are simpler: we are talking only about the cohomology of a polynomial algebra. On the other hand, it is more complicated in the sense that the answer is bigger.

Roughly speaking, when a spectrum has a motivic version defined over $\mathbb{C}$, then we can also consider its reduction to the Gheorghe-Isaksen-Wang-Xu motivic spectrum S^{Mot}/τ, where, by their theorem, the derived Adams-Novikov-type spectral sequence collapses (the *special fiber*). On the other hand, when we invert τ, we get the situation of classical homotopy theory (the *generic fiber*). In the motivic setting over $\mathbb{C}$ (the *mixed characteristic*), the higher differentials of the derived Adams-Novikov-type spectral sequence (whose E_2-term, in our case, is a polynomial algebra with generator τ over the corresponding E_2-term over S^{Mot}/τ), correspond exactly to τ-torsion of the motivic spectrum in question.

Our particular use of the method of Gheorghe, Isaksen, Wang, and Xu [29, 14, 15] is quite different from the way it was applied by Isaksen, Wang and Xu [30] and Burkland, Xu [11]. In our present setting, we study the motivic rectified Adams-Novikov spectral sequence directly, investigating the behavior of τ-torsion in the context of spectral algebra. In [30], on the other hand, the main point is to use the additional fact of coincidence of the Adams spectral sequence for S^{Mot}/τ with the algebraic Novikov spectral sequence, which is then used to deduce facts about the classical Adams spectral sequence differentials.

As far as explicit calculations of (3), due to the complicated nature of formal group laws, a closed form answer is not known. However, for the purpose of practical calculations, there is a suitable filtration on the cobar complex which leads to an algebraic spectral sequence, on whose E_1-term the algebra A considered above acts trivially. This spectral sequence therefore has a parabolic vanishing curve with asymptotic slope 0. Concrete computations using symbolic algebra were carried out by Riley [53]. We will discuss this approach in Section 4 below.

3. The rectified Adams-Novikov spectral sequence for $MO[2]$

Virtually all the spectral sequences used to calculate stable homotopy groups (such as the Adams spectral sequence, the Adams-Novikov spectral sequence, see [50] for a quick review) are spectral sequence of descent type, using the standard construction of Godement [17]. In the present paper, we work in the ∞-category of R-modules where R is an E_∞-ring spectrum over which the complex cobordism spectrum MU is an E_∞-algebra (for foundations, see [38, 13]), where concretely $R = MO[2]$ and $R = MSC$, and the monad (i.e. standard construction) on R-modules is

$$M \mapsto M \wedge_R MU.$$

(Comment: In this paper, by ∞-*category* we mean a derived category with some point-set theoretical underpinning in which we can do topological limits and colimits. This can be accomplished using the quasi-categorical setup of [39], but we are also equally happy to work in the more classical approach of topological categories used, for example, in [38, 13]. For efforts on formally unifying approaches to ∞-categories, see e.g. [7, 52].)

The corresponding descent spectral sequences, converging to $MO[2]_*$, MSC_*, have E_2-term which can be expressed as

$$Ext_\Gamma(A, A) = Cotor_\Gamma(A, A)$$

where the Hopf algebroid (A, Γ) is the Hopf algebroid (6) in the case of $MO[2]$ and (7) in the case of MSC provided that the Hopf algebroid is sufficiently nice (in the sense that Γ is flat over A). We refer to these spectral sequences as the *Rectified Adams-Novikov spectral sequence* for $MO[2]$ resp. MSC.

Additionally, the spectral sequences automatically converge to their target assuming that the connectivity of the iterated fibers increases. Both conditions turn out to be true in our case. However, this requires a calculation of the Hopf algebroids (6), (7).

In the present section, we give a complete algebraic computation of (6), which is simpler. More than a warm-up, the result of the present section is actually a precursor of the computation of the Hopf algebroid for the case of MSC, which is our main point of interest.

3.1. **The Hopf algebroid** (L, LS)**.** It is well known that $MU_* = L$ is the Lazard ring which supports the universal formal group law. Additionally, the Landweber-Novikov Hopf algebroid $(MU_*, MU_*MU) = (L, LB)$ [35, 46] has

$$LB = L[b_1, b_2, \dots]$$

where b_i are thought of as the coefficients of a series

$$b(x) = x + \sum_{i>0} b_i x^{i+1}$$

where $f : F \to G$ represents strict isomorphisms from a given formal group law to another formal group law ([50], Appendix A2).

We shall now describe a Hopf algebroid (L, LS) as follows: Suppose we write

$$b(x) = x + s(x)[2]x \tag{8}$$

where

$$s(x) = \sum_{i>0} s_i x^i,$$

and that we simultaneously impose the relation

$$xi(x) = b(x)b(i(x)) \tag{9}$$

where $i(x) = [-1]x$ is the inverse series. Plugging in (8) into (9), we get

$$s(x)i(x)[2]x + xs(i(x))[-2]x + s(x)s(i(x))[2]x[-2]x = 0. \tag{10}$$

Now clearly $[-2]x = i([2]x)$ is divisible by $[2]x$, so (10) can be further rewritten as

$$s(x)i(x) + xs(i(x))\frac{[-2]x}{[2]x} + s(x)s(i(x))[-2]x = 0. \tag{11}$$

However, we claim that (10) is once more divisible by $[2]x$. To this end, note that

$$\frac{[-2]x}{[2]x} \equiv -1 \mod [2]x,$$

so modulo $[2]x$, (11) has the form

$$s(x)i(x) - xs(i(x)) = s(x)(i(x) - x) + x(s(x) - s(i(x))),$$

which is divisible by $x - i(x)$, which in turn is a unit multiple of

$$x -_F i(x) = [2]x.$$

Thus, denote by

$$g(x) = \sum_{n \geq 1} g_n x^n$$

the ratio of the left hand side of (11) by $[2]x$. Then we have

$$g_{2n} = \pm s_{2n} \mod I \tag{12}$$

where I is the augmentation ideal of $L[s_1, s_2, \dots]$. We will see later using topology that the relations g_{2n+1} actually follow from the relations g_{2n}. Thus, we can define the Hopf algebroid (L, LS) by

$$LS = L[s_{2n+1} \mid n \geq 0] = L[s_n \mid n \geq 1]/(g_{2n} \mid n \geq 1). \tag{13}$$

3.2. **The E_2-term of the rectified Adams-Novikov spectral sequence for $MO[2]$.** We are now ready to state and prove the main result of the present section, identifying the E_2-term of the rectified Adams-Novikov spectral sequence converging to $MO[2]_*$.

2. **Theorem.** *(1) We have $LS \cong MU \wedge_{MO[2]} MU_*$.*

(2) The rectified Adams-Novikov spectral sequence for $MO[2]$ has the form

$$Cotor_{(L,LS)}(L, L) \Rightarrow \pi_* MO[2]. \tag{14}$$

Proof. We begin by computing $MU \wedge_{MO[2]} MU_*$. This is a connective spectrum of finite type, so we can compute one prime at a time. At $p > 2$, we have

$$H^*MO[2] \cong H^*BO \cong \mathbb{Z}/p[p_1, p_2, \dots]$$

so $H^*(MO[2], \mathbb{Z}/p)$ is a direct sum of even suspensions of $P^* = A^*/(\beta)$ by the Milnor-Moore theorem, and hence $MO[2]$ is a wedge of even suspensions of copies of BP, which map by (5) equivalently to wedge summands of MU. We conclude that $MU \wedge_{MO[2]} MU_* = LS$ locally at an odd prime.

To compute at $p = 2$, we first use the Eilenberg-Mac Lane spectral sequence

$$Tor^{H_*MO[2]}(H_*MU, H_*MU) \Rightarrow H_*MU \wedge_{MO[2]} MU. \tag{15}$$

We have

$$H_*(MO[2], \mathbb{Z}/2) \cong H_*(BO, \mathbb{Z}/2) = \mathbb{Z}/2[a_1, a_2, \dots] \tag{16}$$

where $|a_i| = i$, while we can write

$$H_*(MU, \mathbb{Z}/2) = H_*(BU, \mathbb{Z}/2) = \mathbb{Z}/2[a_2, a_4, \dots].$$

Thus, the E^2-term of the spectral sequence (15) is

$$\Lambda_{\mathbb{Z}/2}[b_1, b_3, \dots] \tag{17}$$

where b_{2i+1} has topological dimension $2i+1$ and Tor-dimension 1 and thus, total dimension $2i + 2$. Additionally, since homological suspension preserves Dyer-Lashof operations, by considering the Dyer-Lashof operations in H_*BO ([48]), in homology, (17) becomes

$$\mathbb{Z}/2[b_{4i+1} \mid i \geq 0] \tag{18}$$

where $|b_{4i+1}| = 4i + 2$. Thus, by Milnor-Moore's theorem, again, $H_*(MU \wedge_{MO[2]} MU, \mathbb{Z}/2)$ is a direct sum of even suspensions of P_*, and hence also at 2, $MU_{MO[2]}MU$ is a wedge of even suspensions of copies of BP with the correct number of terms in each degree.

In fact, this is also true multiplicatively, so we know that

$$MU \wedge_{MO[2]} MU_* = MU_*[s_1, s_3, \dots]$$

where $|s_{2i+1}| = 4i + 2$.

Additionaly, considering the canonical map

$$MU_*MU \to MU \wedge_{MO[2]} MU_*, \tag{19}$$

we get from the Bockstein spectral sequence that

$$2s_i = b_i \mod \text{decomposables}. \tag{20}$$

We also now know that (19) is onto rationally and that the target has no torsion.

To identify the elements s_i precisely, we first recall that considering the standard complex orientation $x \in MU^*\mathbb{C}P^\infty$, and identifying it with the element of $(MU \wedge MU)^*\mathbb{C}P^\infty$ by composing with the left unit $\eta_L : MU \to MU \wedge MU$, then composing with the right unit,we obtain the element $b(x) \in (MU \wedge MU)^*\mathbb{C}P^\infty$. Similarly, if we use $\mathbb{R}P^\infty$ instead of $\mathbb{C}P^\infty$, we get the same result modulo $[2]x$. However, the two maps $\mathbb{R}P^\infty \to MU \wedge_{MO[2]} MU$ using the left and right unit must coincide, since the orientation $\mathbb{R}P^\infty \to MU$ factors through $MO[2]$. Thus, we deduce that in $MU \wedge_{MO[2]} MU_*$, $b(x)$ must be congruent to x modulo $[2]x$, and thus, the elements s_i defined by (8) must exist in $MU \wedge_{MO[2]} MU_*$. We further see from (20) that s_{2i+1} coincide with the expected generators (since the leading term of $[2]x$ is $2x$).

To identify the precise relations between these elements, consider the second Chern class

$$c_2 : BU(2) \to \Sigma^4 MU.$$

Composing with the canonical inclusions

$$\mathbb{C}P^\infty \to BO(2) \to BU(2),$$

the second Chern class becomes

$$xi(x)$$

where x is the complex orientation. If we compose with the left resp. right unit to $MU \wedge MU$, we get $xi(x)$, $b(x)b(i(x))$, respectively. However, since the composition factors through $MO[2]$, in $MU \wedge_{MO[2]} MU_*$, (9) must hold. This proves the relations we established. (We can divide $[2]x$ twice since there is no torsion.) Further, it proves that the odd-degree part of (9) can generate no further relation, since we know from the above calculation that the elements s_{2i+1} are algebraically independent.

□

Comment: The question of a moduli interpretation of the Hopf algebroid (L, LS) is a natural one. We completed the calculation by embedding into the rationalization of the Hopf algebroid (L, LB) representing the groupoid of formal group laws and strict isomorphisms. We imposed congruence of the strict isomorphism to the identity modulo the 2-series $[2]_F x$, which caused us to allow some division of the coefficients of the reparametrization series, and then we imposed the condition that the strict isomorphism preserve the series $xi(x)$ (the parameter of the Buchstaber-Novikov 2-valued formal group [8, 9]), which caused the even-degree coefficients to become polynomial functions of the odd-degree ones. Thus, we can say that *LS represents strict isomorphisms which are identity on the 2-torsion of the formal group, and preserve the coordinate of the 2-valued formal group.* However, this should be considered more of a metaphor than a precise statement.

4. The rectified Hopf algebroid for MSC

In Section 3, we calculated the Hopf algebroid (6). In this section, we shall calculate the Hopf algebroid (7). Again, locally at an odd prime, there is nothing to prove, as MSC is just a wedge of copies of BP (see [33, 54]).

4.1. **Computation of** $(MU \wedge_{MSC} MU)_*$**.** At the prime 2, we again need to calculate $(MU \wedge_{MSC} MU)_*$. As in the proof of Theorem 2, we again first calculate $H_*(MU \wedge_{MSC} MU, \mathbb{F}_2)$. To this end, we recall from [33] that we have

$$H_*(MSC, \mathbb{F}_2) = H_*(MU, \mathbb{F}_2) \otimes \Lambda(a_1, a_3, \dots) = \mathbb{F}_2[a_2, a_4, \dots] \otimes \Lambda(a_1, a_3, \dots) \tag{21}$$

where, comparing with (16), the map $H_*(MO[2], \mathbb{F}_2) \to M_*(MSC, \mathbb{F}_2)$ is realized by reduction modulo a_{2i+1}^2. Thus, again, we have a spectral sequence

$$H_*(MU, \mathbb{F}_2) \otimes Tor^{\Lambda(a_1, a_3, \dots)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*(MU \wedge_{MSC} MU, \mathbb{F}_2). \tag{22}$$

Using the homology of an exterior algebra in characteristic 2, the left hand side of (22) is

$$H_*(MU, \mathbb{F}_2) \otimes \Lambda(b_{1,i}, b_{3,i}, \dots, i = 0, 1, \dots) \tag{23}$$

where $b_{2j+1,i}$ is the ith transpotence of b_{2j+1}, and thus has topological dimension $(2j+1) \cdot 2^i$ and Tor-dimension s^i. Therefore, all elements are in even total dimension, and thus, the spectral sequence (22) collapses fo E^2. As in Section 3, using Dyer-Lashof operations, we see that the square of $b_{2j+1,i}$ is represented by $b_{4j+3,i}$, and thus, it is possible to write

$$H_*(MU \wedge_{MSC} MU, \mathbb{F}_2) = H_*(MU, \mathbb{F}_2) \otimes \mathbb{F}_2[b_{4j+1,i} \mid i, j \in \mathbb{N}_0]. \tag{24}$$

From (24) and the Milnor-Moore theorem, we can write

$$H_*(MU \wedge_{MSC} MU, \mathbb{F}_2) = P_* \otimes \mathbb{F}_2[x_i \mid i \neq 2^k] \otimes \mathbb{F}_2[b_{4j+1,i} \mid i, j \in \mathbb{N}_0] \tag{25}$$

and hence the Adams spectral sequence also collapses to E_2 and we can write

$$\pi_*(MU \wedge_{MSC} MU)_2^\wedge = MU_*[b_{4j+1,i}]_2^\wedge. \tag{26}$$

Noticing that the total dimensional degree of $b_{4j+1,i}$ is $(2j+1) \cdot 2^{i+1}$, which runs through all even degrees, comparing to the odd prime computation, and using finite type, we can then simply change notation to write

$$LSC = MU \wedge_{MSC} MU_* = MU_*[b'_1, b'_2, \dots] \tag{27}$$

where the dimensional degree of b'_i is $2i$.

4.2. **Computation of the Hopf algebroid** (L, LSC)**.** To begin our work on identifying the E_2-term of the rectified Adams-Novikov spectral sequence converging to MSC_*, first note that in fact, we can further identify these elements by composing with the Hurewicz homomorphism into

$$H_*(MU \wedge_{MSC} MU, \mathbb{Z}) = H_*(BU \times_{BSC} BU, \mathbb{Z}) \tag{28}$$

(which follows from the Thom isomorphism). We can identify $BU \times_{BSC} BU$ with

$$BU \times BU/BSC \sim BU \times BU \tag{29}$$

where on the left hand side of (29), the second copy of BU is identified with the antidiagonal in $BU \times BU$. The right hand side of (29) follows from the fibration sequence

$$BSC \longrightarrow BU \xrightarrow{1-\overline{?}} BU. \tag{30}$$

Thus, the Thom isomorphism identifies

$$H_*(MU \wedge_{MSC} MU, \mathbb{Z}) = H_*(MU \wedge BU, \mathbb{Z}) \tag{31}$$

where $H_*BU = \mathbb{Z}[b'_1, b'_2, \dots]$ are the usual generators.

We can consider a diagram

$$\begin{array}{ccc} & & LB = MU_*[B_1, B_2, \dots] \\ & \swarrow & \downarrow \\ LS = MU_*[s_1, s_3, \dots] & & \\ & \searrow & \\ & & LSC = MU_*[b_1, b_2, \dots]. \end{array} \tag{32}$$

(We used capital letters in the top term to avoid a confusion between LB and LSC, which are isomorphic graded algebras.) It is convenient to write

$$B(x) = x + B_1x^2 + B_2x^3 + \dots,\ b(x) = x + b_1x^2 + b_2x^3 + \dots,$$

$$\overline{B}(x) = B(x)/x,\ \bar{b}(x) = b(x)/x.$$

Then the vertical map (32) is given by

$$\overline{B}(x) \mapsto \bar{b}(x)/\bar{b}(i(x)) \tag{33}$$

where $i(x)$ is the universal formal inverse. We see that our relation (9) translates to

$$\overline{B}(x)\overline{B}(i(x)) = 1,$$

which holds. Writing

$$B(x) = x + s(x)[2]x,\ \overline{s}(x) = s(x)/x,$$

we get

$$\overline{B}(x) = \overline{s}(x)[2]x + 1.$$

To compute the composition law on (L, LSC), recall that the composition law on (L, LB) is given by

$$\Delta(B(x)) = B_1 \circ B_2(x) \tag{34}$$

where $B_1 = B \otimes 1$, $B_2 = 1 \otimes B$. Thus, we are looking for a composition law on (L, LSC) which would be compatible with the map (33). To this end, we note that (33) gives

$$B(x) \mapsto \frac{b(x)i(x)}{b(i(x))}. \tag{35}$$

When calculating composition, we must keep in mind that the inverse transforms by the right unit in composition. Putting, for a series $g(x) = x + \dots$,

$$i_g(x) = g(i(g^{-1}(x))), \tag{36}$$

using (35), and denoting

$$\Delta(b)(x) = \phi(x),$$

we get by (34)

$$\frac{b_1(g(x))i_g(g(x))}{b_1(i_g(g(x)))} = \frac{\phi(x)}{\phi(i(x))} \cdot i(x) \tag{37}$$

where

$$g(x) = \frac{b_2(x)i(x)}{b_2(i(x))}.$$

Using (36), we get

$$\frac{b_1(\dfrac{b_2(x)i(x)}{b_2(i(x))}) \cdot \dfrac{b_2(i(x))x}{b_2(x)}}{b_1(\dfrac{b_2(i(x))x}{b_2(x)})} = \frac{\phi(x)}{\phi(i(x))} \cdot i(x).$$

This leads to the natural guess

$$\phi(x) = \Delta(b(x)) = b_1(\frac{b_2(x)i(x)}{b_2(i(x))}) \cdot \frac{b_2(i(x))}{i(x)} \tag{38}$$

where

$$b_1 = b \otimes 1,\ b_2 = 1 \otimes b.$$

The unit axiom follows immediately by replacing $b_1(x)$ resp. $b_2(x)$ with x.

To verify associativity, putting

$$b_1 = b\otimes 1\otimes 1,\ b_2 = 1\otimes b\otimes 1,\ b_3 = 1\otimes 1\otimes b,$$
$$z = \frac{b_3(x)i(x)}{b_3(i(x))},\ \overline{z} = \frac{b_3(i(x))x}{b_3(x)},$$

we get

$$(1\otimes\Delta)\Delta b(x) = b_1(\frac{b_2(z)\cdot b_3(i(x))x}{b_2(\overline{z})b_3(x)})\cdot b_2(\overline{z})\cdot\frac{b_3(x)}{xi(x)}$$

while

$$(\Delta\otimes 1)\Delta b(x) = b_1(\frac{b_2(z)\cdot\overline{z}}{b_2(\overline{z})})\cdot\frac{b_2(\overline{z})}{\overline{z}}\cdot\frac{b_3(i(x))}{i(x)}.$$

It is immediate that both expressions are equal.

3. **Theorem.** *The Hopf algebroid structure on (L, LSC) is determined as follows: The left unit is the inclusion, the right unit is the right unit in (L, LB) composed with (35). The augmentation sends $\overline{b}(x)\mapsto 1$, and the coproduct is given by (38).*

Proof. As a warm-up, we begin by reformulating the structure formulas of (L, LB). As seen in [50], Theorem A2.1.16, it suffices to give the structure formulas in the Hopf algebroid

(39) $$(HL, HLB) = (H\mathbb{Z}_*MU, H\mathbb{Z}_*MU\wedge MU).$$

Using our above convention of using a bar to indicate division by x, we get

(40) $$\overline{b_1\circ b_2}(x) = \overline{b}_1(b_2(x))\cdot\overline{b}_2(x).$$

Now if we write

$$b_2 : F\to G,$$

then

$$b_2(exp_F(x)) = exp_G(x),$$

while

$$exp_G(x) = \eta_R(exp_F(x))$$

(where η_R is applied on coefficients). Thus, (40) implies

(41) $$\overline{b_1\circ b_2}(exp_F(x)) = \overline{b}_1(\eta_R exp_F(x))\cdot\overline{b}_2(exp_F(x)).$$

In other words, letting

(42) $$b(exp_F(x)) = x + g_1x^2 + g_2x^3 + \dots,$$

then

(43) $$b'_i\mapsto g_i$$

gives a morphism of Hopf algebroids

$$(\mathbb{Z},\mathbb{Z}[b'_1,b'_2,\dots])\mapsto(HL,HLB)$$

where

$$(\mathbb{Z},\mathbb{Z}[b'_1,b'_2,\dots])=(\mathbb{Z},H_*BU)$$

is the standard Hopf algebra coming from the loop space structure on BU, i.e.

$$\psi(b'_i)=\sum_{j+k=i}b'_j\otimes b'_k.$$

Having derived this formula algebraically, we can also see it geometrically, applying the Thom isomorphism to (39). Of course, saying that this determines the Hopf algebroid structure on (L,LB) would be misleading, since to apply this algebraically, we would first need to have the formula for η_R.

However, putting

$$(HL,HLSC)=(H\mathbb{Z}_*MU,H\mathbb{Z}_*MU\wedge_{MSC}MU),\tag{44}$$

we already know that the right unit is determined by applying the right unit in (L,LB), followed by (33). Thus, applying the same geometric argument, we see that (43), (42) define a morphism of Hopf algebroids

$$(\mathbb{Z},H_*BU)\to(HL,HLSC).\tag{45}$$

Comparing this with our algebraic formula, we see that (38) gives

$$\Delta\bar{b}(x)=\bar{b}_1(\frac{b_2(x)i(x)}{b_2i(x)})\cdot\bar{b}_2(x).$$

Comparing this with (33), we see that our algebraic formula for Δ in $(HL,HLSC)$ agrees with the morphism of Hopf algebroids (45). Since we know the right unit a priori, the formula (38) is proved. To finish proving the statement of our theorem, it suffices to identify the image of LSC in $HLSC$, which, however, follows from our spectral sequence computation, and from analogous consideration at primes $p>2$. □

4.3. **The Witt construction.** To understand better the Hopf algebroid structure of (L,LSC), recall that it suffices to work locally at the prime $p=2$. The advantage of working locally is that we can profit from the theorem by Ravenel and Wilson on the structure of bipolynomial algebras [51, 28].

We begin by describing the following generalization of a construction of Husemoller [28]. Suppose (A, R) is a Hopf algebroid. An element $s \in R$ is called *primitive* if

$$\Delta(s) = 1 \otimes s + s \otimes 1 := (\eta_L \otimes Id + Id \otimes \eta_R)(s). \tag{46}$$

Note that a primitive element represents a class in

$$Cotor^1_{(A,R)}(A, A) = Ext^1_{(A,R)}(A, A). \tag{47}$$

4. **Definition.** *Let (A, R) be a Hopf algebroid and let $S \subseteq R$ be a set of primitive elements such that $R = R_0[S]$ for some A-algebra R. Then the* Witt construction *$(A, W_S(R))$ is defined by*

$$W_S(R) = R_0[S \times \mathbb{N}_0] = R_0[s_i \mid s \in S, i \in \mathbb{N}_0]$$

where $\Delta(s_i)$ is determined by requiring that the "ghost component"

$$p^i s_i + p^{i-1} s_{i-1}^p + \cdots + p s_1^{p^{i-1}} + s_0^{p^i}$$

be primitive. (Note: as usual, these elements are to be interpreted by using the universal formulas which they imply in the absence of $\mathbb{Z}$-torsion.)

5. **Lemma.** *The Hopf algebroid $(A, W_S(R))$, up to isomorphism, only depends on the images of the elements $s \in S$ in (47).*

Proof. The Witt construction is a pullback of a diagram of affine groupoid schemes

$$\begin{array}{ccc} (X, \Phi) & \xrightarrow{f} & (\bullet, \mathbb{G}_a) \\ & & \uparrow \pi \\ & & (\bullet, G). \end{array} \tag{48}$$

Changing the representatives of the cohomology classes corresponds to choosing a morphism of affine schemes

$$w : X \to \mathbb{G}_a$$

and replacing f by g given on

$$\alpha : x \to y$$

by

$$g(\alpha) = f(\alpha) + w(x) - w(y)$$

where the addition denotes the operation in $\mathbb{G}_a$. Therefore, the conclusion of the Lemma holds if we can lift w to G:

$$\begin{array}{ccc} X & \xrightarrow{w} & \mathbb{G}_a \\ & \overline{w} \searrow & \uparrow \pi \\ & & G. \end{array}$$

The existence of lifting in our case follows from the fact that π is the *Spec* of the unit of a polynomial algebra. □

6. **Theorem.** *Let $S = \{s_1, s_3, \dots\} \subset LS$ be the elements represented by real projective spaces $\mathbb{R}P^{4i+1}$, $i \in \mathbb{N}_0$. Then, locally at 2, we have a canonical isomorphism of Hopf algebroids*

$$(L, LSC) \cong (L, W_S LS). \tag{49}$$

Proof. Since the Witt construction Hopf algebroid is commutative and generated by elements s where $\Delta(s)$ does not involve any elements of MU_* of dimensional degree > 0, it is given by a coaction of a Hopf algebra on a comodule algebra. Furthermore, this Hopf algebra is bipolynomial. Similar conclusions apply also to the Hopf algebroid (L, LSC) (see the proof of Theorem 3). Thus, we may apply Proposition 2.3 of Ravenel and Wilson [51]. □

Comment: It is worth noting now that our spectral sequences converging to $MO[2]$, MSC, despite being based on resolutions by MU-modules, are *not* the same as the Adams-Novikov spectral sequences for these spectra. A way to see this is to consider the generator $a_1 \in MSC_1 = MO[2]_1$, which corresponds to

$$(\overline{s}_1) \in Ext^1_{LSC}(MU_*, MU_*).$$

We see from (16) that applying the Hurewicz homomorpism

$$\pi_* MO[2] \to MU_* MSC \to H\mathbb{Z}/2_* MO[2], \tag{50}$$

the class a_1 goes to the class a_1. (This map is given, in fact, by taking the first Stiefel-Whitney class of the specified 1/2 of the stable normal bundle of $\mathbb{R}P^1$, which is the Möbius strip.) Thus, the class a_1 in the source of (50), which is 4-torsion, survives all the maps, and hence the middle term of (50) must have 4-torsion (the case of MSC is the same). We conclude that the Adams-Novikov cobar complexes for MSC, $MO[2]$ have torsion, while the rectified cobar complex does not.

$s\backslash t-s$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	$\mathbb{Z}$		0		$\mathbb{Z}$		0		$\mathbb{Z}^2$		0		$\mathbb{Z}^3$
1		(4)		$\mathbb{Z}$		$(2,16)$		$\mathbb{Z}^2$		$(8,64)$		$\mathbb{Z}^4$	
2			0		(2)		(4)		$(2,4,8)$		$(\mathbb{Z},2,4)$		$(2,4^2,8,32)$
3				0		0		0		(2)		(2)	
4					0		0		0		0		0

FIGURE 1. Self-conjugate cobordism groups

Comment: The Witt construction can be described as a polynomial coalgebra on generators in topological degrees $2j$ for $j = 1, 2, \dots$. Thus, the E_2-term of the rectified Adams-Novikov spectral sequence for MSC can be described as (2).

On the other hand, there is a decreasing filtration on the Witt construction where for every primitive generator s, the iterated Verschiebung s_i is 2^i. This leads to an algebraic spectral sequence whose E_1-term is

$$Ext_A(\mathbb{Z}, \mathbb{Z}) \otimes MU_* \tag{51}$$

where the polynomial generators of A act on $\mathbb{Z}$ trivially (which is forced by dimensional degree). This leads to an analog of the May spectral sequence, which only has non-zero terms for

$$t - s \geq s^2$$

For calculations of MSC_*, see Figure 1 - the numbers indicate orders of cyclic summands; thus, for example, the group in dimension $t-s = 12$, $s = 2$ is $\mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/32$.

For the E_2-term of the algebraic spectral sequence (51), see Figure 2. Subscripts of entries indicate their algebraic filtration degrees. We can see from the table that the algebraic spectral sequence has both higher differentials and extensions. For example, there is a d_2 from $(t-s, s) = (5, 1)$ to $(t-s, s) = (4, 2)$. There is an extension in $(t-s, s) = (7, 1)$.

5. THE COLLAPSE OF THE RECTIFIED ADAMS-NOVIKOV SPECTRAL SEQUENCE FOR MSC

In this section, we prove Theorem 1. The proof has two parts. Frst, we make an observation about invertibility of modules over an E_∞-ring spectrum which arises in the very special situation when we have a convergent spectral sequence of the form (2). We then bring in techniques of motivic homotopy theory to give the concrete argument in our situation.

$s \setminus t-s$	0	1	2	3	4	5	6	7	8
0	$\mathbb{Z}_0$		0		$\mathbb{Z}_0$		0		$\mathbb{Z}_0^2$
1		$(4)_1$		$\mathbb{Z}_2$		$(4_1, 16_1)$		$(2_1, \mathbb{Z}_2, \mathbb{Z}_4)$	
2			0		(4_3)		(4_2)		$(4_3, 16_3, 4_5)$
3				0		0			

$s \setminus t-s$	9	10	11	12
0		0		$\mathbb{Z}_0^3$
1	$(2_1^2, 8_1, 64_1)$		$(2_1^2, \mathbb{Z}_2^2, \mathbb{Z}_4, \mathbb{Z}_8)$	
2		$(2_1, 4_1, 2_3, \mathbb{Z}_6)$		$(2_3^2, 8_3, 64_3, 4_5, 16_5, 4_3)$
3	(4_4)		(4_7)	

FIGURE 2. The algebraic rectified Adams-Novikov spectral sequence for MSC

5.1. **Invertible modules.** We begin with a general observation. Let R be an E_∞ ring spectrum and let $\alpha_1, \dots, \alpha_n, \dots \in R_*$ be elements, and let M be an R-module. We are interested in the example

$$R = MSC,\ M = MU. \tag{52}$$

Then we can form an R-module

$$F_{(a_1,\dots,a_n,\dots)}(R) = \operatorname*{holim}_n \Sigma^{1-n} R/(\alpha_1, \dots, \alpha_n). \tag{53}$$

In fact, we can similarly form

$$F_{(a_1,\dots,a_n,\dots)}(M) = \operatorname*{holim}_n \Sigma^{1-n} M \wedge_R R/(\alpha_1, \dots, \alpha_n). \tag{54}$$

The comparison

$$M \wedge_R F_{(a_1,\dots,a_n,\dots)}(R) \xrightarrow{\sim} F_{(a_1,\dots,a_n,\dots)}(M) \tag{55}$$

is a matter of convergence, although the map always exists canonically and (55) holds in the case of (52). In our present setting, convergence holds due to increasing connectivity of maps between the fibers

$$\Sigma^{-n} R/(\alpha_1, \dots, \alpha_{n+1}) \to \Sigma^{1-n} R/(\alpha_1, \dots, \alpha_n).$$

Now note further that in the case (52), $F_{(a_1,\dots,a_n,\dots)}(M)$ maps into our cobar MSC-resolution with a map inducing an isomorphism on E_2-terms. (This simply follows from the fact that the elements a_i are permanent cycles, a known fact which is recalled in the Appendix.) Thus, we have

$$F_{(a_1,\dots,a_n,\dots)}(M) \sim R. \tag{56}$$

Together with (55), this implies, in fact, that $F_{(a_1,\dots,a_n,\dots)}(R)$ is a strong dual of M in the derived category DR of R-modules, and that in fact, more strongly, both objects are invertible and inverse to each other.

5.2. **Motivic homotopy theory.** To see how this implies the collapse of our spectral sequence, we need to recall some more context, namely the setting of motivic homotopy theory of Morel and Voevodsky [42]. This is a technique for studying homotopy theory of certain schemes using *motivic spaces*, which means simplicial Nisnevich sheaves of sets. One takes the simplicial homotopy category, localized at projections of the form $X \times \mathbb{A}^1 \to X$. The theory works well for separated smooth schemes of finite type over a field, where calculations can be made, culminating with Voevodsky's solution of the Bloch-Kato conjecture [56].

There is also a category of motivic spectra (also described in [42]), which arises by stabilizing the ∞-category of based (=pointed) motivic spaces with respect to smash-product with the simplicial S^1-sphere, as well as the multiplicative group $\mathbb{G}_m$. A key point is that

$$S^1 \wedge \mathbb{G}_m \sim \mathbb{P}^1,$$

so one may equivalently stabilize with respect to the 1-dimensional projective space $\mathbb{P}^1$. There is also a corresponding theory of E_∞-ring spectra, and hence spectral algebra [31, 23].

The ∞-category of topological spaces is equivalent to the ∞-category of simplicial sets, which are motivic spaces. For this reason, every ordinary topological spectrum has an avatar in the category of motivic spectra. However, in cases where the spectrum comes from some type of geometrical construction, a more natural algebraic version often presents itself. This is in particular the case of ordinary homology H with coefficients $\mathbb{Z}$, complex K-theory K, and complex cobordism MU, whose appropriate algebraic versions represent motivic cohomology, algebraic K-theory of smooth schemes, and algebraic Thom cobordism MGL [42, 37].

Specifically in the case of MGL, one builds the universal motivic Thom space $BGL(n)^{\gamma_n}$ of n-bundles by taking the colimit over N of the one-point compactification of the total space of the tautological bundle of the Grassmannian variety of n-vector subspaces in $\mathbb{A}^{n+N}$. One has a canonical map

$$\mathbb{P}^1 \wedge BGL(n)^{\gamma_n} \to BGL(n+1)^{\gamma_{n+1}},$$

thus giving rise to the algebraic cobordism spectrum MGL. A more careful version of this construction also gives an E_∞-ring spectrum [31].

It is noteworthy from our point of view that due to fact that the map (1) also has a natural algebraic version, we have, by a similar construction, a motivic version MSC^{Mot} of the spectrum MSC. (For the purposes of this paper, we shall only work in the 2-complete motivic category over the field $\mathbb{C}$, suppressing the completion from the notation.)

Defining the motivic category by Morel and Voevodsky [42] was an important milestone, but the definition itself gives little hint why we could calculate anything using this method. An algebraic topologist might say that we can calculate quite a bit if we know the cohomology of a point. Motivically, we typically don't, but due to Voevodsky's fundamental theorem [56], we know the *ℓ-completed cohomology of a point*. This gives a path to obtaining information about motivic spectra arising from constructions of rational algebraic geometry, which includes algebraic K-theory as well as MGL.

When Morel and Voevodsky first developed their theory, the focus was on obtaining information about the underlying field, whose ℓ-completed motivic cohomology is trivial in the algebraically closed case. The idea of studying motivic homotopy theory over $\mathbb{C}$ was first considered in [27, 26]. The full significance of this was only recently understood and developed into an important new tool of stable homotopy theory by Gheorghe, Isaksen, Wang, and Xu [29, 14, 15, 30], with a boom of further developments by multiple authors, see e.g. [2, 10, 16, 49, 11, 6].

Back to our proof, in [26], Section 4, it was calculated that

$$MGL_\star = MU_*[\tau] \tag{57}$$

where the generators x_i, τ have dimensions $i(1+\alpha)$, $(1-\alpha)$ in the notation of [26], where the element τ was denoted θ. For general background on algebraic cobordism, we refer the reader to Levine and Morel [37]. The "$1,\alpha$" notation is motivated by analogs with $\mathbb{Z}/2$-equivariant homotopy theory via the Real realization - the analogy was noticed in the 1990's by Hu and Kriz, who used it in several subsequent papers. In recognition of the connections with algebraic geometry, it has become more common to denote the dimensions by $1=(1,0)$, $\alpha=(1,1)$.

Now the constructions of the present paper may be repeated verbatim in the 2-completed motivic category over $\mathbb{C}$, we obtain a variant of the spectral sequence (2) of the form

$$Ext_A(\mathbb{Z}, MU_*)[\tau] \Rightarrow MSC_*^{Mot}. \tag{58}$$

Now we can use the result of Gheorghe [14] (see also [15, 30]) which asserts that when we change rings from S^{Mot} to S^{Mot}/τ, the resulting spectral sequence

$$Ext_A(\mathbb{Z}, MU_*) \Rightarrow (MSC^{Mot}/\tau)_* \tag{59}$$

collapses.

In fact, any higher differentials in (58) give rise to τ-torsion in MSC^{Mot}_*. Now let us return to the setup of the beginning of this section, this time putting

$$R = MSC^{Mot},\ M = MGL \tag{60}$$

(still working in the 2-completed motivic category over $\mathbb{C}$). As above, we conclude again that $M = MGL$ is an invertible object in the derived category of R-modules. By (57), its homotopy groups have no τ-torsion. Suppose now $0 \neq b \in \pi_* R$ were τ-torsion. Therefore, the element b would have to act by 0 on M. However, since M is invertible, it would therefore also act by 0 on R, which is a contradiction.

The lack of $\mathbb{Z}$-multiplicative extensions is proved similarly: Suppose

$$2^m x = y\tau \tag{61}$$

for $x, y \in R_*$. Then the relation (61) will also be true in the corresponding operations on the invertible module M_*. However, in our case, $M_* = MU_*[\tau]$, whose operations are $MU^* MU[\tau]$ and thus, (61) does not occur.

Comment: We do not know if the spectral sequence

$$Ext_{LS}(L, L) \Rightarrow MO[2]_* \tag{62}$$

collapses to the E_2-term. However, recall that the primitive generators $\overline{s}_{2k+1}$ of degrees $(t-s, s) = (4k+1, 1)$ are permanent cycles (represented by $\mathbb{R}P^{4k+1}$). Now these manifolds all have non-zero Stiefel-Whitney numbers of the half-normal number. Equivalently, they produce a nontrivial image by the Hurewicz homomorphism into $H\mathbb{Z}/2_*(MO[2])$. Hence, all powers of these generators are non-zero (in contrast, for example, with Nishida's nilpotence theorem in the stable homotopy groups of spheres). See Figure 3.

6. Appendix: The classical methods

The subject of MSC was extensively studied, see for example [8, 12, 18, 36, 43, 44, 54]. We recall here some known partial results some of which are implicit in our discussion, and which are not easily quotable in the literature, at least in the present context.

$s\backslash\ t-s$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	$\mathbb{Z}$		0		$\mathbb{Z}$		0		$\mathbb{Z}^2$		0		$\mathbb{Z}^3$
1		(4)		0		$(2,16)$		0		$(8,64)$		0	
2			(2)		0		$(2,4)$		(2)		$(2,4^2)$		(4)
3				(2)		0		(2^2)		(2)		(2^5)	
4					(2)		0		(2^2)		(2)		(2^5)
5						(2)		0		(2^2)		(2)	
6							(2)		0		(2^2)		(2)
7								(2)		0		(2^2)	
8									(2)		0		(2^2)
9										(2)		0	
10											(2)		0
11												(2)	
12													(2)

FIGURE 3. The rectified Adams-Novikov spectral sequence for $MO[2]$

Let us denote by $\mathcal{L}$ the subring of all elements $x \in MU_*$ such that

$$c_{2i+1}c_{j_1}\dots c_{j_\ell}[x] = 0$$

for all $i, j_1 \dots, j_\ell \in \mathbb{N}$.

7. **Theorem.** *The ring $\mathcal{L}$ coincides with the image of the canonical map $\iota : MSC_* \to MU_*$.*

To prove this, note that $Im(\iota) \subseteq \mathcal{L}$ was proved by Buchstaber [8], Lemma 24.17. We also have

6.1. **Proposition.** *(Buchstaber [8], Theorem 24.20) If we denote by $\kappa : MU_* \to MO_*$ the canonical map, then*

$$Im(\kappa\iota) = \kappa(\mathcal{L}). \tag{63}$$

□

6.2. **Proposition.** *Let*

$$\beta_n = b_n^2 - 2b_{n-1}b_{n+1} + \dots + 2(-1)^n b_{2n} \in MU_* \otimes \mathbb{Q}$$

where

$$b(x) = x + \sum_{n\geq 1} b_n x^{n+1}$$

is the exponential series of the universal formal group law. Then

$$\mathcal{L} \otimes \mathbb{Q} = \mathbb{Q}[\beta_1, \beta_2, \dots]. \tag{64}$$

Proof. The series

$$\beta(x) = -x^2 + \sum_{n\geq 1} \beta_n (-x^2)^{n+1}$$

satisfies

$$\beta(x) = b(x)b(-x) = b(x)ib(x)$$

where $i(x)$ is the formal inverse. Thus, considering

$$b : \mathbb{C}P^\infty \to \Sigma^2 MU_\mathbb{Q},\ \beta : \mathbb{C}P^\infty \to \Sigma^4 MU_\mathbb{Q},$$

we can write

$$\beta = b\bar{b}$$

where $\bar{b}$ denotes complex conjugation. In other words, β can be expressed as the composition

$$\mathbb{C}P^\infty \xrightarrow{\phi} BU(2) \xrightarrow{c_2} \Sigma^4 MU_\mathbb{Q}$$

where c_2 is the Conner-Floyd Chern class and ϕ is B applied to the embedding

$$S^1 \to U(2)$$

by

$$z \mapsto \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}.$$

Thus, the map κ factors as

$$\mathbb{C}P^\infty \to BO(2) \to BU(2)$$

where the second map is complexification. Therefore, $\beta_n \in \mathcal{L} \otimes \mathbb{Q}$ by the fact that rationally, odd Chern classes vanish on a complexified real bundle. On the other hand, it follows from considering rational homology that $Im(\iota) \otimes \mathbb{Q}$ is a polynomial algebra on generators in dimensions divisible by 4 (for example by Conner-Floyd [12]), and thus our statement follows from a counting argument. □

Recall the Milnor class $s_n = p_n(c_1, c_2, \dots, c_n)$ in Chern classes where

$$p_n(\sigma_1, \sigma_2, \dots) = t_1^n + t_2^n + \dots$$

where σ_i are the elementary symmetric polynomials in the t_i. Recall that the Milnor number $s_n[x]$ detects the image of an element $x \in MU_{2n}$ in the module of indecomposables QMU_{2n}.

6.3. **Proposition.** *There exists an element $V_k \in Im(\iota)_{2(2^k-1)}$ whose Milnor number is 8. These elements are equal to v_k^2 (where we denote $v_k = x_{2^k-1}$) modulo other monomials in the x_i. Addionally, V_k can be chosen so that $\kappa(V_k) = 0$.*

Proof. For the first statement, it suffices to construct an element in the given dimension with Milnor number of 2-valuation 3 (since at odd primes, MSC is just a wedge of copies of BP, see [12]). Now for $k \geq 3$, one notes that

$$v_2\binom{2^{k+1}+2^k-4}{2^{k+1}-7} = 3.$$

It follows that in this case, we can take a Stong manifold given as the $\mathbb{Z}/2$-quotient of an intersection of 2^k-2 hypersurfaces of bidegree $(1,1)$ in

$$\mathbb{C}P^{2^{k+1}-7} \times \mathbb{C}P^{2^k+3}$$

by the diagonal $\mathbb{Z}/2$-involution on both $\mathbb{C}P^{2i+1}$-factors:

$$(z_0, z_1, \dots, z_{2i}, z_{2i+1}) \mapsto (-\overline{z_1}, \overline{z_0}, \dots, -\overline{z_{2i+1}}, \overline{z_{2i}}).$$

For $k = 1, 2$, one must use other generators (e.g. [43] observes that the statement of Theorem 7 is true in dimensions ≤ 128).

Now by Proposition 6.2, V_k is congruent to $4\beta_{2^k-1}$ modulo the square of the augmentation ideal in $Im(\iota) \otimes \mathbb{Q}$. We see that no multiples of the monomials of $\beta_i\beta_j$ contain v_k^2, which is a monomial summand of $4\beta_{2^k-1}$. The second statement follows. Finally, for the last statement, by [44, 8], $Im(\kappa\iota)$ is the 4th power of the Floyd ring. In particular, it is a polynomial ring with generators in dimensions $8(2i+1)$, $8 \cdot 2^\ell$, $8 \cdot i$ where i is not a power of 2. Therefore, lifting the generators to $Im(\iota)$, none of them can contain a rational multiple of β_{2^k-1} as a summand (for reasons of dimension). Therefore, adding a polynomial in these generators cannot cancel the term v_k^2. □

Proof of Theorem 7. It remains to prove that

$$\mathcal{L} \subseteq Im(\iota). \tag{65}$$

Since (as we already noted) the problem is trivial at odd primes, we may work completed at 2. Thus, suppose $y \in \mathcal{L}_2^\wedge$. By Proposition 6.2, we may write

$$y = p(V_1, V_2, \dots)$$

where p is a polynomial with coefficients in

$$\mathbb{Q}_2[x_i \mid i \neq 2^k-1].$$

However, the coefficient a_ℓ where $\ell = (\ell_1, \ell_2, \dots)$ of $V_1^{\ell_1}V_2^{\ell_2}\dots$ must in fact satisfy

$$a_\ell \in \mathbb{Z}_2[x_i \mid i \neq 2^k-1],$$

since otherwise the element y would not belong to $(MU_*)_2^\wedge$ (consider the coefficient of $v_1^{2\ell_1}v_2^{2\ell_2}\dots$). Additionally, we must also have $a_\ell \in$

$\mathbb{Q}_2[\beta_1, \beta_2, \dots]$ (since the element of highest degree which fails this condition would contradict Proposition 6.2).

Now by Proposition 6.1, there exist $b_\ell \in Im(\iota)$ so that

$$b_\ell - a_\ell \in (2, v_1, v_2, \dots).$$

Now since we also have $b_\ell - a_\ell \in \mathcal{L}$, it will have a lower degree (and hence be subject to induction) except when $\ell = 0$. However, the constant term of $b_0 - a_0$ is now divisible by 2. Thus, we may divide the constant term by 2 and apply the same procedure to it, and apply induction to its other coefficients. Since we are working completed at 2, the infinite sum in increasing powers of 2 we produce by repeating this process will converge to an element of $Im(\iota)^\wedge_2$ which is equal to y.

□

Suppose now we filter LSC by 1/2 times the topological degree of the L-degree of the augmentation $LSC \to L$. This is a decreasing filtration, and one has an algebraic Novikov spectral sequence

$$E_2 = Cotor_{(L, E_0 LSC)}(L, L) \Rightarrow Cotor_{(L, LSC)}(L, L). \tag{66}$$

Moreover, it follows from the discussion of the previous section that the left hand side of (66) is of the form

$$Cotor_{(L, E_0 LSC)}(L, L) = \Lambda_L(a_1, a_3, a_5, \dots) \tag{67}$$

where the generator a_{2k+1} is in degree $2k+1$. Moreover, it follows from considering the Adams spectral sequence that the generators a_{4k+1} are realized by the manifolds $\mathbb{R}P^{4k+1}$ whose stable tangent bundle is $(4k+2)\gamma^1_{\mathbb{R}}$, which is double a real bundle, and thus canonically has a structure of a self-conjugate complex bundle.

On the other hand, the representatives N^{4k-1} of a_{4k-1} were constructed by Landweber [36, 54]. They are given by

$$S^{4k-1} \times_{Sp(1)} S^3$$

where $Sp(1)$ acts on S^{4k-1} (thought of as the unit sphere in $\mathbb{H}^k$) by right multiplication of quaternions, and on $S^3 = Sp(1)$ by conjugation. (Here we are considering only the compact form of $Sp(k)$.)

One then remarks that the sum of the tangent bundle of N^{4k-1} and a 1-dimensional trivial real bundle is isomorphic to $k\gamma^1_{\mathbb{H}}$, and thus has a canonical structure of a self-conjugate complex bundle.

6.4. **Proposition.** *The Toda brackets*

$$\langle a_{2k+1}, a_{2k+1}, \dots, a_{2k+1} \rangle \tag{68}$$

all contain $0 \in MSC_*$.

Proof. We begin by showing that

(69) $$a_{2k+1}^2 = 0.$$

When k is even, consider the manifold with boundary

$$M = \mathbb{R}P^{4k+1} \times \mathbb{R}P^{4k+1} \times [0,1]$$

with $\mathbb{Z}/2$-action by

(70) $$(x,y,t) \mapsto (y,x,1-t).$$

The fixed point submanifold is

$$\Delta \times \{1/2\}$$

where $\Delta \subset \mathbb{R}P^{4k+1} \times \mathbb{R}P^{4k+1}$ is the diagonal. Thus, the normal bundle

$$\nu_\Delta^M = (4k+2)\gamma_{\mathbb{R}}^1,$$

with $\mathbb{Z}/2$-action by

(71) $$x \mapsto -x.$$

This is canonically a complex bundle, so we can perform a complex blow-up of Δ in N and form a non-singular manifold Z by taking the $\mathbb{Z}/2$-quotient (since the submanifold of $\mathbb{Z}/2$-fixed points now has complex codimension 1). Moreover, the construction just performed is complex and self-conjugate, thus proving that the manifold Z with boundary has an MSC-structure, thus providing the cobordism which proves (69).

In the case of k odd, we put, analogously,

$$M = N^{4k-1} \times N^{4k-1} \times [0,1],$$

again with $\mathbb{Z}/2$-action by (70). This time, the fixed point manifold is

$$E \times \{1/2\}$$

where $E \subset N^{4k-1} \times N^{4k-1}$ is the diagonal. Thus, the normal bundle is

$$\nu_E^M = k\gamma_{\mathbb{H}}^1.$$

Once again, this is naturally a complex bundle, so we can perform a complex blow-up of E, and then take a $\mathbb{Z}/2$-quotient, thus again getting a manifold with boundary Z. Since, again, the construction performed is complex and self-conjugate, Z is an MSC-cobordism, again proving (69).

Now assume an MSC-cobordism Z_n is constructed proving (67) with n factors. If k is even, we form a manifold M_n by gluing $Z_n \times \mathbb{R}P^{4k+1}$ and $\mathbb{R}P^{4k+1}$ along

$$\mathbb{R}P^{4k+1} \times M_{n-1} \times \mathbb{R}P^{4k+1}$$

and multiplying by $[0,1]$. The manifold M_n has a natural action by reversing the order of the copies of $\mathbb{R}P^{4k+1}$ (and extending by the corresponding maps on the cobordism coordinates), and mapping

$$t \mapsto 1-t$$

on the new interval coordinate. The action is free on the previous cobordism coordinates, so the fixed point manifold D_n is $\{1/2\}$ times the fixed point of the $\mathbb{Z}/2$-action on

$$(\mathbb{R}P^{4k+1})^n$$

by reversing the order of factors. This is a diagonal manifold isomorphic to $(\mathbb{R}P^{4k+1})^{\lceil n/2 \rceil}$, and its normal bundle is a sum of $\lfloor n/2 \rfloor$ copies of the $(4k+2)\gamma^1_{\mathbb{R}}$ on the individual coordinates, and a trivial bundle, with $\mathbb{Z}/2$-action by (71). (Because of the previous cobordism coordinates, there are always enough trivial coordinates to stabilize.) Thus, the normal bundle of D_n in M_n is a self-conjugate complex bundle, and we can again perform a complex blow-up of D_n in M_n, and then take a $\mathbb{Z}/2$-quotient. The construction is complex and self-conjugate, and thus, we obtain the required cobordism proving (67) with $n+1$ factors.

The case of k odd is completely analogous, with $\mathbb{R}P^{4k+1}$ replaced by N^{4k-1}. □

7. Statements and Declarations

The authors have no competing interests.

References

[1] D. W. Anderson: The real K-theory of classifying spaces, *Proc. Nat. Acad. Sci.* 51 (1964) 634-636

[2] G. Angelini-Knoll, M. Behrens, E. Belmont, H.J. Kong: A deformation of Borel equivariant homotopy, https://arxiv.org/abs/2308.01873

[3] S. Araki: τ-cohomology theories, *Japan Journal of Mathematics* (N.S.) 4 (2) (1978) 363-416, Forgetful spectral sequences, *Osaka Journal of Mathematics* 16 (1) (1979) 173-199, Orientations in q-cohomology theories, *Japan Journal of Mathematics* (N.S.) 5 (2) (1979) 403-430.

[4] M. F. Atiyah: *K -theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1967

[5] M. F. Atiyah: K-theory and reality, *Quart. J. Math. Oxford Ser.* (2) 17 (1966), 367-386

[6] T. Bachmann, H. J. Kong, G. Wang, Z. Xu: The Chow t-structure on the ∞-category of motivic spectra, *Ann. of Math.* (2) 195 (2022), no.2, 707-773

[7] J. Bergner: *The homotopy theory of* $(\infty,1)$*-categories*, London Math. Soc. Student Texts 90. Cambridge Univ. Press, 2018, 284 pp.

[8] V. M. Buchstaber: Characteristic classes in cobordism and topological applications of the theory of single-valued and two-valued formal groups, *Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat.* Volume 10, (1978) 5-178

[9] V. M. Buchstaber, S. P. Novikov: Formal groups, power systems and Adams operators, *Mat. Sb.* (N.S.) 84 (126) (1971), 81-118

[10] R. Burklund, J. Hahn, A. Senger: Galois reconstruction of Artin-Tate R-motivic spectra, 2020. arXiv:2010.10325

[11] R. Burkland, Z. Xu: The Adams differentials on the classes h_j^3, arXiv:2302.11869

[12] P. E. Conner, E. E. Floyd: Torsion in SU-bordism, *Mem. Amer. Math. Soc.* 60 (1966)

[13] A. Elmendorf, I. Kriz, M. Mandell and J. P. May: *Rings, Modules and Algebras in Stable Homotopy Theory*, AMS Surveys and Monographs series 47, American Mathematical Society, 1997

[14] B. Gheorghe: The Motivic Cofiber of τ, *Doc. Math.* 23 (2018), 1077–1127

[15] B. Gheorghe, G. Wang, and Z. Xu: The special fiber of the motivic deformation of the stable homotopy category is algebraic, *Acta Math.*, 226(2) (2021) 319-407

[16] B. Gheorghe, D. C. Isaksen, A. Krause, N. Ricka: C-motivic modular forms. *J. Eur. Math. Soc.* 24(10) (2022) 3597-3628

[17] R. Godement: Topologie Algèbrique et Théorie des Faisceaux, *Actualités Sci. Ind.* No. 1252. Publ. Math. Univ. Strasbourg. No. 13 Hermann, Paris 1958

[18] N. Y. Gozman: On the image of the self-conjugate cobordism ring in the complex and unoriented cobordism rings, *Dokl. Akad. Nauk SSSR*, Volume 216, Number 6, (1974), 1212-1214

[19] P. S. Green: A cohomology theory based upon self-conjugacies of complex vector bundles, *Bull. AMS* 70 (1964) 522-524

[20] V. K. A. M. Gugenheim, J. P. May: *On the theory and applications of differential torsion products*, Memoirs of the American Mathematical Society, No. 142. American Mathematical Society, Providence, R.I., 1974

[21] M. A. Hill, M. J. Hopkins, D. C. Ravenel: On the nonexistence of elements of Kervaire invariant one, *Ann. of Math.* (2) 184 (2016), no. 1, 1-262

[22] M. Hovey, B. Shipley, J. Smith.: Symmetric spectra, *J. Amer. Math. Soc.* 13 (2000), 149-208

[23] P. Hu: S -modules in the category of schemes, *Mem. Amer. Math. Soc.* 161 (2003), no.767, viii+125 pp.

[24] P. Hu: The cobordism of Real manifolds, *Fund. Math.* 161 (1999), no. 1-2, 119-136

[25] P. Hu and I. Kriz: Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence, *Topology* 40 (2001), no. 2, 317-399

[26] P. Hu, I. Kriz, K. Ormsby: Remarks on motivic homotopy theory over algebraically closed fields, *J. K-Theory* 7 (2011), no. 1, 55-89

[27] P. Hu, I. Kriz, and K. Ormsby: The homotopy limit problem for Hermitian K-theory, equivariant motivic homotopy theory and motivic Real cobordism, *Advances in Mathematics* 228 (2011), no. 1, 434–480

[28] D. Husemoller: The structure of the Hopf algebra $H^*(BU)$ over a $\mathbb{Z}_{(p)}$-algebra, *Am. J. of Math.* 43, (1971), 329–349

[29] D. C. Isaksen: Stable stems. *Mem. Amer. Math. Soc.* 262 (2019), no.1269, viii+159 pp.

[30] D. C. Isaksen, G. Wang, Z. Xu: Stable homotopy groups of spheres: from dimension 0 to 90.(English summary), *Publ. Math. Inst. Hautes Études Sci.* 137(2023), 107-243

[31] J. F. Jardine: Motivic symmetric spectra, *Doc. Math.* 5(2000), 445-552

[32] N. Kitchloo, W. S. Wilson: The $ER(n)$-cohomology of $BO(q)$ and real Johnson-Wilson orientations for vector bundles, *Bulletin of the London Math. Soc.*, 47 no. 5 (2015), 835-847

[33] A. Kono, M. Nagata: Homology of $(KSC)_n$ over the Dyer-Lashof algebra, *Japan. J. Math.* 6 (1980) 45-60

[34] P. S. Landweber: Conjugations on complex manifolds and equivariant homotopy of MU_*, *Bulletin of the American Mathematical Society* 74 (1968) 271-274

[35] P. S. Landweber: Cobordism operations and Hopf algebras, *Trans. Amer. Math. Soc.* 129 (1967), 94-110

[36] P. S. Landweber: On the symplectic bordism groups of the spaces $Sp(n)$, $HP(n)$ and $BSp(n)$, *Michigan Math.* J. 15 (1968), 145–153

[37] M. Levine, F. Morel: Cobordisme algébrique I, *C. R. Acad. Sci. Paris Sér. I Math.* 332 (2001), no. 8, 723-728

[38] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure: *Equivariant stable homotopy theory*, Vol 1213 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1986

[39] J. Lurie. Higher algebra. Harvard University, 2017
https://www.math.ias.edu/∼lurie/papers/HA.pdf

[40] M. Mandell, J. P. May, S. Schewede and B. Shipley: Model categories of diagram spectra. *Proc. London Math. Soc.* 82 (2001) 441-512

[41] J. W .Milnor: On the cobordism ring Ω_* and a complex analogue, *Amer. J. Math.* 82 (1960), 505-521

[42] F. Morel, V. Voevodsky: A^1-homotopy theory of schemes, *Inst. Hautes Etudes Sci. Publ. Math.*, 90 (1999) 45-143

[43] R. Nadiradze: Characteristic classes in SC-theory and their applications, *Proceedings of A. Razmadze Mathematical Institute*, Volume 104, (1995) 55-74

[44] R. Nadiradze: Involutions on Stong manifolds and their applications in cobordism theory, *Russ. Math. Surv.* 35 (1980) 264

[45] S. P. Novikov: Some problems in the topology of manifolds connected with the theory of Thom spaces, *Dokl. Akad. Nauk. SSSR.* 132 (1960), 1031-1034

[46] S. P. Novikov: The methods of algebraic topology from the viewpoint of cobordism theories, *Izv. Akad. Nauk. SSSR. Ser. Mat.* 31 (1967), 855-951

[47] S. P. Novikov: Homotopy properties of Thom complexes, *Mat. Sb.* (N.S.) 57 (99) (1962), 407-442

[48] S. Priddy: Dyer-Lashof operations for the classifying spaces of certain matrix groups, *Quart. J. Math.* Oxford Ser. (2), 26(102) (1975) 179–193

[49] P. Pstragowski: Synthetic spectra and the cellular motivic category. *Invent. Math.*, 232(2) (2023) 553- 681

[50] D. C. Ravenel: *Complex cobordism and stable homotopy groups of spheres*, AMS Chelsea Publishing, 2003

[51] D. C. Ravenel, W. S. Wilson: Bipolynomial Hopf Algebras, *Journal P. Applied Alg.* 4 (1974) 41-45

[52] E. Riehl, D. Verity: *Elements of ∞-category theory*, Cambridge Studies in Advanced Mathematics 194, Cambridge University Press, Cambridge, 2022. xix+759 pp.

[53] B. Riley: *Self-Conjugate Cobordism and the Rectified Adams-Novikov Spectral Sequence*, A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Mathematics), University of Michigan, 2024

[54] L. Smith, R. E. Stong: The structure of BSC, *Inventionnes Math.* 5 (1968) 138-159

[55] R. Thom: Quelques propriétés globales des varietés differentiables, *Comm. Math. Helv.* 28 (1954), 109-181

[56] V. Voevodsky: On motivic cohomology with $\mathbb{Z}/\ell$-coefficients. *Ann. of Math.* (2) 174 (2011), no. 1, 401-438