NEW VARIANTS OF THE ADAMS-NOVIKOV SPECTRAL SEQUENCE AND AN ALGEBRAIC COMPUTATION OF SELF-CONJUGATE COBORDISM

PO HU, IGOR KRIZ, BENJAMIN RILEY AND PETR SOMBERG

Abstract. We give a complete algebraic computation of self-conjugate cobordism groups MSC_*, which has been an open problem since the 1960's. Our approach is based on several new ideas, including structured homotopy theory, new apsects of formal group laws and variants of the Adams-Novikov spectral sequence, and the motivic loop of Gheorghe, Isaksen, Wang and Xu. We also obtain new results on cobordism with antilinear involution.

1. Introduction

The problem of computing the ring of complex cobordism groups MU_*, i.e. the complex version of Thom's problem [36], was solved in 1960 by Milnor [26] and Novikov [29, 31]. The *complex cobordism spectrum* MU, along with Atiyah's *K-theory* [2], were two generalized cohomology theories which played a key role in the development of modern algebraic topology.

Studying the effect of *complex conjugation* on these theories was a next logical step. This can be done in at least two different ways: It is possible to consider *self-conjugate structures*, which are complex structures (e.g. manifolds or bundles) equipped with an antilinear automorphism. On the other hand, one can make an antilinaer involution a part of the structure and study it $\mathbb{Z}/2$-*equivariantly.* This is often referred to as a *Real* structure.

In the case of K-theory, the two approaches are related in a relatively simple way. Both Real and self-conjugate K-theory $K\mathbb{R}$ and KSC were introduced by Atiyah in his elegant proof of real Bott periodicity [3]. Self-conjugate K-theory KSC turns out to be a part of the equivariant Real K-theory $K\mathbb{R}$, (or, speaking non-equivariantly, a shift of orthogonal K-theory KO), thus explaining it completely.

The authors acknowledge support by grants GA CR P201/12/G028 and GA CR 19-28628X. Benjamin Riley was supported by NSF RTG grant DMS 1840234, Hu was supported by NSF grant DMS-2301520, Kriz also acknowledges the support of a Simons Collaboration Grant.

In cobordism, the situation is rather different. The $\mathbb{Z}/2$-equivariant *Real cobordism spectrum* $M\mathbb{R}$ was defined by Landweber [21], and further investigated by Araki [1]. The coefficient ring $M\mathbb{R}_\star$ was eventually calculated by Hu and Kriz [14], and later used, along with its variants, by Hill, Hopkins and Ravenel [12] in their solution of the Kervaire-Milnor problem. It is worth noting that due to the behavior of equivariant transversality, the equivariant Thom spectrum $M\mathbb{R}$ does not actually calculate the corresponding cobordism groups of manifolds, which were characterized by Hu [13] as homotopy groups of suspension spectra, making them essentially uncomputable by current methods.

The spectrum MSC of *self-conjugate cobordism*, on the other hand, was studied by Smith and Stong [35]. It is defined as the Thom spectrum of the pullback of the tautological element of K^0BGL to BSC, which is defined as the homotopy equalizer

(1)
$$BSC \longrightarrow BGL \underset{Id}{\overset{A\mapsto(A^T)[^{-1}}{\rightrightarrows}} BGL$$

(BGL is homotopically equivalent to BU, but we write BGL to emphasize the fact that the construction makes sense algebraically, which will be relevant below.)

Despite the analogy with KSC, the spectrum MSC is *not* a part of the $\mathbb{Z}/2$-equivariant Real cobordism spectrum $M\mathbb{R}$. In fact, MSC turns out to be much more complicated than $M\mathbb{R}$. For example, the group MSC_1 is $\mathbb{Z}/4$ [10]. To give another example, the image of the map $MSC_4 \to MU_4$ is generated by an Enriques surface (by computation of Chern numbers). The homotopy groups of the spectrum MSC, which are the geometrically defined cobordism groups of manifolds with self-conjugate complex structures on their normal bundles, resisted calculation until the present time.[1]

The goal of this paper is to introduce a completely algebraic machine (using the homological algebra of formal groups) which gives a complete calculation of MSC_*, even though the answer is too complicated to be written out in closed form.

Concretely, using formal group law theory, we construct an explicit action of a certain polynomial algebra $A = \mathbb{Z}[\alpha_1, \alpha_2, \alpha_3, \dots]$ on MU_* where α_i has topological degree $-2i$ (graded homologically), and a spectral sequence

(2)
$$E_2^{s,t} = Ext_A^{s,t}(\mathbb{Z}, MU_*) \Rightarrow MSC_*.$$

[1] These groups have no p-torsion for $p \neq 2$, and their localization at $p \neq 2$ was calculated by Smith and Stong [35].

Our main result is

1. **Theorem.** *The spectral sequence (2) collapses and there are no $\mathbb{Z}$-multiplicative extensions. In other words, the self-conjugate cobordism groups are given by*

$$MSC_n = \bigoplus_{t-s=n} Ext_A^{s,t}(\mathbb{Z}, MU_*). \tag{3}$$

Our proof of Theorem 1 is based on several new techniques which were not available at the time when MSC was first considered.

1.1. **Spectral commutative rings.** In the construction of our spectral sequence, we use *strictly commutative spectral rings* (i.e. E_∞-ring spectra [25, 7]) In that category, we have a morphism

$$MSC \to MU,$$

and our spectral sequence is obtained by resolving MU over

$$MU \wedge_{MSC} MU.$$

The resulting *Hopf algebroid* (i.e. coordinate ring of an affine groupoid scheme) has a surprisingly simple description, and (at least 2-locally, which is the only non-trivial prime in this case), has the form outlined above.

In fact, to make these conclusions, we also study the spectrum $MO[2]$, which represents the cobordism of manifolds M with a real bundle ν and an isomorphism

$$\tau_M \oplus \nu \oplus \nu \cong N$$

where N is a trivial bundle. Then $MO[2]$ can be described as the spectrum associated with the prespectrum (D_{2n}) where

$$D_{2n} = BO(n)^{2\gamma_{\mathbb{R}}^n} \tag{4}$$

and the connecting map $\Sigma^2 D_{2n} \to D_{2n+2}$ given by the obvious isomorphism of the pullback of $2\gamma_{\mathbb{R}}^{n+1}$ to $BO(n)$ with $2\gamma_{\mathbb{R}}^n \oplus 2$. We refer to $MO[2]$ as the *double real cobordism spectrum*. This spectrum was studied before, although not extensively (see for example Kitchloo and Wilson [19]).

To connect with MSC, note that one can think of $BO(n)$ as the classifying space of complex n-bundles with an antilinear involution, so (4) can also be characterized as $BO(n)^{\gamma_{\mathbb{C}}^n}$ where $\gamma_{\mathbb{C}}^n$ is the universal complex n-bundle. On the other hand, $BSC(n)$ is the classifying space of complex n-bundle with an antilinear automorphism which is not necessarily an involution. Thus, MSC is defined in the same way, with (4) replaced by $BSC(n)^{\gamma_{\mathbb{C}}^n}$.

Complexification $BO(n) \to BU(n)$ therefore induces a canonical morphisms of E_∞-ring spectra

$$MO[2] \to MSC \to MU. \tag{5}$$

We first calculate the simpler Hopf algebroid $(MU_*, MU \wedge_{MO[2]} MU_*)$, which is a precursor for the spectral sequence (2).

1.2. **Rectified Novikov spectral sequences.** The spectral sequences for calculating MSC_* and $MO[2]_*$ are similar to the Adams-Novikov spectral sequence in the sense that they use a resolution by free MU-modules, but the cobar complex is not the same as the Adams-Novikov cobar complex. This behavior arises due to the fact that MU_*MSC, $MU_*MO[2]$ are not flat MU_*-modules.

We call our spectral sequence *rectified Adams-Novikov spectral sequences* because their E_2-terms are described as Ext-groups of particularly nice new Hopf algebroids

$$(MU_*, (MU \wedge_{MO[2]} MU)_*) = (L, LS) \tag{6}$$

and

$$(MU_*, (MU \wedge_{MSC} MU)_*) = (L, LSC). \tag{7}$$

Finding a purely algebraic description of those Hopf algebroids is a key part of our method. The description is based on carefully studying the formal group law structures involved. While the idea of using additional structure on formal group laws in the investigation of MSC is not new (notably, Buchstaber and Novikov [4, 5] used their concept of 2-*valued formal group laws* in this context), the precise description of the structures represented by our novel Hopf algebroids are in fact quite subtle (see Comment at the end of Section 2). Calculating (6) leads to a calculation of (7) using the Witt construction and structure results on bipolynomial Hopf algebras [34].

In the case of MSC, the particularly simple E_2-term then allows an argument proving collapse. The situation is similar to the paper by Gugenheim and May [11], although the present situation is homotopical, not homological, and therefore methods of spectral algebra, which are quite different from the methods of differential graded algebra used in [11], are required.

While the calculation of the Hopf algebroid (6) is a crucial step in calculating the Hopf algebroid (7), the E_2-term of the resulting spectral sequence for $MO[2]$ is more complicated than for MSC, and we do not have a collapse theorem in that case. It is worth noting that we do not really know any differentials in the case of $MO[2]$, and certain surprising elements survive (see Comment at the end of Section 4). Therefore, the possibility of collapse of the rectified Novikov spectral sequence for $MO[2]$ remains an interesting question.

1.3. **The motivic loop and Novikov formality.** The last ingredient in the proof of Theorem 1 is the very recent *motivic loop* technique of Gheorghe, Isaksen, Wang and Xu [8, 18]. A striking complete algebraic calculation of the stable homotopy groups of a space was accomplished by Gheorghe [8], who described the homotopy groups of the $\mathbb{C}$-based motivic spectrum S^{Mot}/τ as the E_2-term of the classical Adams-Novikov spectral sequence. Theorem 1 identifies MSC as another example of a spectrum form which some version of a Novikov resolution gives a complete calculation of homotopy groups. We suggest a term *Novikov-formal spectra* for such examples, even though we do not give a precise definition.

While Gheorghe's theory concerns *motivic spectra*, it has revolutionized the computations of classical stable homotopy groups. Isaksen, Wang and Xu [18] used Gheorghe's result to greatly expand the known calculations of stable homotopy groups of spheres. The case of MSC is somewhat different: it is simpler in the sense that the algebraic objects involved are simpler: we are talking only about the cohomology of a polynomial algebra. On the other hand, it is more complicated in the sense that the answer is bigger.

Roughly speaking, when a spectrum has a motivic version defined over $\mathbb{C}$, then we can also consider its reduction to the Gheorghe motivic spectrum S^{Mot}/τ, where, by his theorem, the Novikov-type spectral sequence collapses (the *special fiber*). On the other hand, when we invert τ, we get the situation of classical homotopy theory (the *generic fiber*). In the motivic setting over $\mathbb{C}$ (the *mixed characteristic*), the higher differentials of the Novikov-type spectral sequence (whose E_2-term, in our case, is a polynomial algebra with generator τ over the corresponding E_2-term over S^{Mot}/τ), correspond exactly to τ-torsion of the motivic spectrum in question. This happens both over the sphere and in our rectified Adams-Novikov spectral sequence for MSC.

Our particular use of the method of Gheorghe [8] is quite different from the way it was applied by Isaksen, Wang and Xu [18]. In our present setting, we study the motivic rectified Novikov spectral sequence directly, investigating the behavior of τ-torsion in the context of spectral algebra. In [18], on the other hand, the main point is to use the additional fact of coincidence of the Adams spectral sequence for S^{Mot}/τ with the algebraic Novikov spectral sequence, which is then used to deduce facts about the classical Adams spectral sequence differentials.

As far as explicit calculations of (3), due to the complicated nature of formal group laws, a closed form answer is not known. However, for the purpose of practical calculations, there is a suitable filtration on the cobar complex which leads to an algebraic spectral sequence, in whose E_1-term the algebra A considered above acts trivially. This spectral sequence therefore has a parabolic vanishing curve with asymptotic slope 0. Concrete computations using symbolic algebra were carried out by Riley. We will discuss this approach in Section 3 below.

The present paper is organized as follows: In Section 2, we describe the Hopf algebroid in the case of $MO[2]$. In Section 3, we discuss the case of the Hopf algebroid for MSC. In Section 4, we prove Theorem 1. Finally, in the Appendix, we recall some classical results which have been obtained by previous authors, and which are implicit in our discussion.

2. The rectified Adams-Novikov spectral sequence for $MO[2]$

Virtually all the spectral sequences used to calculate stable homotopy groups (such as the Adams spectral sequence, the Adams-Novikov spectral sequence, see [33] for a quick review) are spectral sequence of descent type, using the standard construction of Godement [9]. In the present paper, we work in the category of R-modules where R is an E_∞-ring spectrum over which the complex cobordism spectrum MU is an E_∞-algebra (for foundations, see [25, 7]), where concretely $R = MO[2]$ and $R = MSC$, and the monad (i.e. standard construction) on R-modules is

$$M \mapsto M \wedge_R MU.$$

The corresponding descent spectral sequences, converging to $MO[2]_*$, MSC_*, have E_2-term which can be expressed as

$$Ext_\Gamma(A, A) = Cotor_\Gamma(A, A)$$

where the Hopf algebroid (A, Γ) is the Hopf algebroid (6) in the case of $MO[2]$ and (7) in the case of MSC provided that the Hopf algebroid is sufficiently nice (in the sense that Γ is flat over A). We refer to these spectral sequences as the *Rectified Adams-Novikov spectral sequence* for $MO[2]$ resp. MSC.

Additionally, the spectral sequences automatically converge to their target assuming that the connectivity of the iterated fibers increases. Both conditions turn out to be true in our case. However, this requires a calculation of the Hopf algebroids (6), (7). In the present section, we give a complete algebraic computation of (6), which is simpler.

It is well known that $MU_* = L$ is the Lazard ring which supports the universal formal group law. Additionally, the Landweber-Novikov Hopf algebroid $(MU_*, MU_*MU) = (L, LB)$ [22, 30] has

$$LB = L[b_1, b_2, \dots]$$

where b_i are thought of as the coefficients of a series

$$b(x) = x + \sum_{i>0} b_i x^{i+1}$$

where $f : F \to G$ represents strict isomorphisms from a given formal group law to another formal group law ([33], Appendix A2).

We shall now describe a Hopf algebroid (L, LS) as follows: Suppose we write

$$b(x) = x + s(x)[2]x \tag{8}$$

where

$$s(x) = \sum_{i>0} s_i x^i,$$

and that we simultaneously impose the relation

$$xi(x) = b(x)b(i(x)) \tag{9}$$

where $i(x) = [-1]x$ is the inverse series. Plugging in (8) into (9), we get

$$s(x)i(x)[2]x + xs(i(x)[-2]x + s(x)s(i(x))[2]x[-2]x = 0. \tag{10}$$

Now clearly $[-2]x = i([2]x)$ is divisible by $[2]x$, so (10) can be further rewritten as

$$s(x)i(x) + xs(i(x))\frac{[-2]x}{[2]x} + s(x)s(i(x))[-2]x = 0. \tag{11}$$

However, we claim that (10) is once more divisible by $[2]x$. To this end, note that

$$\frac{[-2]x}{[2]x} \equiv -1 \mod [2]x,$$

so modulo $[2]x$, (11) has the form

$$s(x)i(x) - xs(i(x)) = s(x)(i(x) - x) + x(s(x) - s(i(x)),$$

which is divisible by $x - i(x)$, which in turn is a unit multiple of

$$x -_F i(x) = [2]x.$$

Thus, denote by

$$g(x) = \sum_{n \geq 1} g_n x^n.$$

the ratio of the left hand side of (11) by $[2]x$. Then we have

$$g_{2n} = \pm s_{2n} \mod I \tag{12}$$

where I is the augmentation ideal of $L[s_1, s_2, \dots]$. We will see later using topology that the relations g_{2n+1} actually follow from the relations g_{2n}. Thus, we can define the Hopf algebroid (L, LS) by

$$LS = L[s_{2n+1} \mid n \geq 0] = L[s_n \mid n \geq 1]/(g_{2n} \mid n \geq 1). \tag{13}$$

2. **Theorem.** *(1) We have $LS \cong MU \wedge_{MO[2]} MU_*$.*

(2) The rectified Adams-Novikov spectral sequence for $MO[2]$ has the form

$$Cotor_{(L,LS)}(L, L) \Rightarrow \pi_* MO[2]. \tag{14}$$

Proof. We begin by computing $MU \wedge_{MO[2]} MU_*$. This is a connective spectrum of finite type, so we can compute one prime at a time. At $p > 2$, we have

$$H^* MO[2] \cong H^* BO \cong \mathbb{Z}/p[p_1, p_2, \dots]$$

so $H^*(MO[2], \mathbb{Z}/p)$ is a direct sum of even suspensions of $P^* = A^*/(\beta)$ by the Milnor-Moore theorem, and hence $MO[2]$ is a wedge of even suspensions of copies of BP, which map by (5) equivalently to wedge summands of MU. We conclude that $MU \wedge_{MO[2]} MU_* = LS$ locally at an odd prime.

To compute at $p = 2$, we first use the Eilenberg-Mac Lane spectral sequence

$$Tor^{H_* MO[2]}(H_* MU, H_* MU) \Rightarrow H_* MU \wedge_{MO[2]} MU. \tag{15}$$

We have

$$H_*(MO[2], \mathbb{Z}/2) \cong H_*(BO, \mathbb{Z}/2) = \mathbb{Z}/2[a_1, a_2, \dots] \tag{16}$$

where $|a_i| = i$, while we can write

$$H_*(MU, \mathbb{Z}/2) = H_*(BU, \mathbb{Z}/2) = \mathbb{Z}/2[a_2, a_4, \dots].$$

Thus, the E^2-term of the spectral sequence (15) is

$$\Lambda_{\mathbb{Z}/2}[b_1, b_3, \dots] \tag{17}$$

where b_{2i+1} has topological dimension $2i+1$ and Tor-dimension 1 and thus, total dimension $2i+2$. Additionally, since homological suspension preserves Dyer-Lashof operations, by considering the Dyer-Lashof operations in H_*BO ([32]), in homology, (17) becomes

$$\mathbb{Z}/2[b_{4i+1} \mid i \geq 0] \tag{18}$$

where $|b_{4i+1}| = 4i+2$. Thus, by Milnor-Moore's theorem, again, $H_*(MU \wedge_{MO[2]} MU, \mathbb{Z}/2)$ is a direct sum of even suspensions of P_*, and hence also at 2, $MU_{MO[2]}MU$ is a wedge of even suspensions of copies of BP with the correct number of terms in each degree.

In fact, this is also true multiplicatively, so we know that

$$MU \wedge_{MO[2]} MU_* = MU_*[s_1, s_3, \dots]$$

where $|s_{2i+1}| = 4i+2$.

Additionaly, considering the canonical map

$$MU_*MU \to MU \wedge_{MO[2]} MU_*, \tag{19}$$

we get from the Bockstein spectral sequence that

$$2s_i = b_i \mod \text{decomposables}. \tag{20}$$

We also now know that (19) is onto rationally and that the target has no torsion.

To identify the elements s_i precisely, we first recall that considering the standard complex orientation $x \in MU^*\mathbb{C}P^\infty$, and identifying it with the element of $(MU \wedge MU)^*\mathbb{C}P^\infty$ by composing with the left unit $\eta_L : MU \to MU \wedge MU$, then composing with the right unit,we obtain the element $b(x) \in (MU \wedge MU)^*\mathbb{C}P^\infty$. Similarly, if we use $\mathbb{R}P^\infty$ instead of $\mathbb{C}P^\infty$, we get the same result modulo $[2]x$. However, the two maps $\mathbb{R}P^\infty \to MU \wedge_{MO[2]} MU$ using the left and right unit must coincide, since the orientation $\mathbb{R}P^\infty \to MU$ factors through $MO[2]$. Thus, we deduce that in $MU \wedge_{MO[2]} MU_*$, $b(x)$ must be congruent to x modulo $[2]x$, and thus, the elements s_i defined by (8) must exist in $MU \wedge_{MO[2]} MU_*$. We further see from (20) that s_{2i+1} coincide with the expected generators (since the leading term of $[2]x$ is $2x$).

To identify the precise relations between these elements, consider the second Chern class

$$c_2 : BU(2) \to \Sigma^4 MU.$$

Composing with the canonical inclusions

$$\mathbb{C}P^\infty \to BO(2) \to BU(2),$$

the second Chern class becomes

$$xi(x)$$

where x is the complex orientation. If we compose with the left resp. right unit to $MU \wedge MU$, we get $xi(x)$, $b(x)b(i(x))$, respectively. However, since the composition factors through $MO[2]$, in $MU \wedge_{MO[2]} MU_*$, (9) must hold. This proves the relations we established. (We can divide $[2]x$ twice since there is no torsion.) Further, it proves that the odd-degree part of (9) can generate no further relation, since we know from the above calculation that the elements s_{2i+1} are algebraically independent.

□

Comment: The question of a moduli interpretation of the Hopf algebroid (L, LS) is a natural one. We completed the calculation by embedding into the rationalization of the Hopf algebroid (L, LB) representing the groupoid of formal group laws and strict isomorphisms. We imposed congruence of the strict isomorphism to the identity modulo the 2-series $[2]_F x$, which caused us to allow some division of the coefficients of the reparametrization series, and then we imposed the condition that the strict isomorphism preserve the series $xi(x)$ (the parameter of the Buchstaber-Novikov 2-valued formal group [4, 5]), which caused the even-degree coefficients to become polynomial functions of the odd-degree ones. Thus, we can say that *LS represents strict isomorphisms which are identity on the 2-torsion of the formal group, and preserve the coordinate of the 2-valued formal group.* However, this should be considered more of a metaphor than a precise statement.

3. The rectified Hopf algebroid for MSC

In Section 2, we calculated the Hopf algebroid (6). In this section, we shall calculate the Hopf algebroid (7). Again, locally at an odd prime, there is nothing to prove, as MSC is just a wedge of copies of BP (see [20, 35]).

At the prime 2, we again first calculate $H_*(MU \wedge_{MSC} MU, \mathbb{F}_2)$. To this end, we recall from [20] that we have

$$(21)\qquad \begin{array}{l} H_*(MSC, \mathbb{F}_2) = H_*(MU, \mathbb{F}_2) \otimes \Lambda(a_1, a_3, \dots) = \\ \mathbb{F}_2[a_2, a_4, \dots] \otimes \Lambda(a_1, a_3, \dots) \end{array}$$

where, comparing with (16), the map $H_*(MO[2], \mathbb{F}_2) \to M_*(MSC, \mathbb{F}_2)$ is realized by reduction modulo a_{2i+1}^2. Thus, again, we have a spectral sequence

$$(22)\quad H_*(MU, \mathbb{F}_2) \otimes Tor^{\Lambda(a_1, a_3, \dots)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*(MU \wedge_{MSC} MU, \mathbb{F}_2).$$

Using the homology of an exterior algebra in characteristic 2, the left hand side of (22) is

$$H_*(MU, \mathbb{F}_2) \otimes \Lambda(b_{1,i}, b_{3,i}, \dots, i = 0, 1, \dots) \tag{23}$$

where $b_{2j+1,i}$ is the ith transpotence of b_{2j+1}, and thus has topological dimension $(2j+1)\cdot 2^i$ and Tor-dimension s^i. Therefore, all elements are in even total dimension, and thus, the spectral sequence (22) collapses fo E^2. As in Section 2, using Dyer-Lashof operations, we see that the square of $b_{2j+1,i}$ is represented by $b_{4j+3,i}$, and thus, it is possible to write

$$H_*(MU \wedge_{MSC} MU, \mathbb{F}_2) = H_*(MU, \mathbb{F}_2) \otimes \mathbb{F}_2[b_{4j+1,i} \mid i, j \in \mathbb{N}_0]. \tag{24}$$

From (24) and the Milnor-Moore theorem, we can write

$$H_*(MU \wedge_{MSC} MU, \mathbb{F}_2) = P_* \otimes \mathbb{F}_2[x_i \mid i \neq 2^k] \otimes \mathbb{F}_2[b_{4j+1,i} \mid i, j \in \mathbb{N}_0] \tag{25}$$

and hence the Adams spectral sequence also collapses to E_2 and we can write

$$\pi_*(MU \wedge_{MSC} MU)^\wedge_2 = MU_*[b_{4j+1,i}]^\wedge_2. \tag{26}$$

Noticing that the total dimensional degree of $b_{4j+1,i}$ is $(2j+1)\cdot 2^{i+1}$, which runs through all even degrees, comparing to the odd prime computation, and using finite type, we can then simply change notation to write

$$LSC = MU \wedge_{MSC} MU_* = MU_*[b'_1, b'_2, \dots] \tag{27}$$

where the dimensional degree of b'_i is $2i$.

In fact, we can further identify these elements by composing with the Hurewicz homomorphism into

$$H_*(MU \wedge_{MSC} MU, \mathbb{Z}) = H_*(BU \times_{BSC} BU, \mathbb{Z}) \tag{28}$$

(which follows from the Thom isomorphism). We can identify $BU \times_{BSC} BU$ with

$$BU \times BU/BSC \sim BU \times BU \tag{29}$$

where on the left hand side of (29), the second copy of BU is identified with the antidiagonal in $BU \times BU$. The right hand side of (29) follows from the fibration sequence

$$BSC \longrightarrow BU \xrightarrow{1-\overline{?}} BU. \tag{30}$$

Thus, the Thom isomorphism identifies

$$H_*(MU \wedge_{MSC} MU, \mathbb{Z}) = H_*(MU \wedge BU, \mathbb{Z}) \tag{31}$$

where $H_*BU = \mathbb{Z}[b'_1, b'_2, \dots]$ are the usual generators.

We can consider a diagram

$$\begin{array}{ccc} & & LB = MU_*[B_1, B_2, \dots] \\ & \swarrow & \downarrow \\ LS = MU_*[s_1, s_3, \dots] & & \\ & \searrow & \downarrow \\ & & LSC = MU_*[b_1, b_2, \dots]. \end{array} \tag{32}$$

(We used capital letters in the top term to avoid a confusion between LB and LSC, which are isomorphic graded algebras.) It is convenient to write

$$B(x) = x + B_1x^2 + B_2x^3 + \dots,\ b(x) = x + b_1x^2 + b_2x^3 + \dots,$$

$$\overline{B}(x) = B(x)/x,\ \overline{b}(x) = b(x)/x.$$

Then the vertical map (32) is given by

$$\overline{B}(x) \mapsto \overline{b}(x)/\overline{b}(i(x)) \tag{33}$$

where $i(x)$ is the universal formal inverse. We see that our relation (9) translates to

$$\overline{B}(x)\overline{B}(i(x)) = 1,$$

which holds. Writing

$$B(x) = x + s(x)[2]x,\ \overline{s}(x) = s(x)/x,$$

we get

$$\overline{B}(x) = \overline{s}(x)[2]x + 1.$$

To compute the composition law on (L, LSC), recall that the composition law on (L, LB) is given by

$$\Delta(B(x)) = B_1 \circ B_2(x) \tag{34}$$

where $B_1 = B \otimes 1$, $B_2 = 1 \otimes B$. Thus, we are looking for a composition law on (L, LSC) which would be compatible with the map (33). To this end, we note that (33) gives

$$B(x) \mapsto \frac{b(x)i(x)}{b(i(x))}. \tag{35}$$

When calculating composition, we must keep in mind that the inverse transforms by the right unit in composition. Putting, for a series $g(x) = x + \dots$,

$$i_g(x) = g(i(g^{-1}(x))), \tag{36}$$

using (35), and denoting

$$\Delta(b)(x) = \phi(x),$$

we get by (34)

$$\frac{b_1(g(x))i_g(g(x))}{b_1(i_g(g(x)))} = \frac{\phi(x)}{\phi(i(x))} \cdot i(x) \tag{37}$$

where

$$g(x) = \frac{b_2(x)i(x)}{b_2(i(x))}.$$

Using (36), we get

$$\frac{b_1(\frac{b_2(x)i(x)}{b_2(i(x))}) \cdot \frac{b_2(i(x))x}{b_2(x)}}{b_1(\frac{b_2(i(x))x}{b_2(x)})} = \frac{\phi(x)}{\phi(i(x))} \cdot i(x).$$

This leads to the natural guess

$$\phi(x) = \Delta(b(x)) = b_1(\frac{b_2(x)i(x)}{b_2(i(x))}) \cdot \frac{b_2(i(x))}{i(x)} \tag{38}$$

where

$$b_1 = b \otimes 1,\ b_2 = 1 \otimes b.$$

The unit axiom follows immediately by replacing $b_1(x)$ resp. $b_2(x)$ with x.

To verify associativity, putting

$$b_1 = b \otimes 1 \otimes 1,\ b_2 = 1 \otimes b \otimes 1,\ b_3 = 1 \otimes 1 \otimes b,$$

$$z = \frac{b_3(x)i(x)}{b_3(i(x))},\ \overline{z} = \frac{b_3(i(x))x}{b_3(x)},$$

we get

$$(1 \otimes \Delta)\Delta b(x) = b_1(\frac{b_2(z) \cdot b_3(i(x))x}{b_2(\overline{z})b_3(x)}) \cdot b_2(\overline{z}) \cdot \frac{b_3(x)}{xi(x)}$$

while

$$(\Delta \otimes 1)\Delta b(x) = b_1(\frac{b_2(z) \cdot \overline{z}}{b_2(\overline{z})}) \cdot \frac{b_2(\overline{z})}{\overline{z}} \cdot \frac{b_3(i(x))}{i(x)}.$$

It is immediate that both expressions are equal.

3. **Theorem.** *The Hopf algebroid structure on (L, LSC) is determined as follows: The left unit is the inclusion, the right unit is the right unit in (L, LB) composed with (35). The augmentation sends $\overline{b}(x) \mapsto 1$, and the coproduct is given by (38).*

Proof. As a warm-up, we begin by reformulating the structure formulas of (L, LB). As seen in [33], Theorem A2.1.16, it suffices to give the structure formulas in the Hopf algebroid

$$(HL, HLB) = (H\mathbb{Z}_*MU, H\mathbb{Z}_*MU \wedge MU). \tag{39}$$

Using our above convention of using a bar to indicate division by x, we get

$$\overline{b_1 \circ b_2}(x) = \overline{b}_1(b_2(x)) \cdot \overline{b}_2(x). \tag{40}$$

Now if we write

$$b_2 : F \to G,$$

then

$$b_2(exp_F(x)) = exp_G(x),$$

while

$$exp_G(x) = \eta_R(exp_F(x))$$

(where η_R is applied on coefficients). Thus, (40) implies

$$\overline{b_1 \circ b_2}(exp_F(x)) = \overline{b}_1(\eta_R exp_F(x)) \cdot \overline{b}_2(exp_F(x)). \tag{41}$$

In other words, letting

$$b(exp_F(x)) = x + g_1x^2 + g_2x^3 + \dots, \tag{42}$$

then

$$b'_i \mapsto g_i \tag{43}$$

gives a morphism of Hopf algebroids

$$(\mathbb{Z}, \mathbb{Z}[b'_1, b'_2, \dots]) \mapsto (HL, HLB)$$

where

$$(\mathbb{Z}, \mathbb{Z}[b'_1, b'_2, \dots]) = (\mathbb{Z}, H_*BU)$$

is the standard Hopf algebra coming from the loop space structure on BU, i.e.

$$\psi(b'_i) = \sum_{j+k=i} b'_j \otimes b'_k.$$

Having derived this formula algebraically, we can also see it geometrically, applying the Thom isomorphism to (39). Of course, saying that this determines the Hopf algebroid structure on (L, LB) would be misleading, since to apply this algebraically, we would first need to have the formula for η_R.

However, putting

$$(HL, HLSC) = (H\mathbb{Z}_*MU, H\mathbb{Z}_*MU \wedge_{MSC} MU), \tag{44}$$

we already know that the right unit is determined by applying the right unit in (L, LB), followed by (33). Thus, applying the same geometric argument, we see that (43), (42) define a morphism of Hopf algebroids

$$(\mathbb{Z}, H_*BU) \to (HL, HLSC). \tag{45}$$

Comparing this with our algebraic formula, we see that (38) gives

$$\Delta\bar{b}(x) = \bar{b}_1(\frac{b_2(x)i(x)}{b_2 i(x)}) \cdot \bar{b}_2(x).$$

Comparing this with (33), we see that our algebraic formula for Δ in $(HL, HLSC)$ agrees with the morphism of Hopf algebroids (45). Since we know the right unit a priori, the formula (38) is proved. To finish proving the statement of our theorem, it suffices to identify the image of LSC in $HLSC$, which, however, follows from our spectral sequence computation, and from analogous consideration at primes $p > 2$. □

To understand better the Hopf algebroid structure of (L, LSC), we present the following generalization of a construction of Husemoller [17]. Suppose (A, R) is a Hopf algebroid. An element $s \in R$ is called *primitive* if

$$\Delta(s) = 1 \otimes s + s \otimes 1 := (\eta_L \otimes Id + Id \otimes \eta_R)(s). \tag{46}$$

Note that a primitive element represents a class in

$$Cotor^1_{(A,R)}(A, A) = Ext^1_{(A,R)}(A, A). \tag{47}$$

4. **Definition.** *Let (A, R) be a Hopf algebroid and let $S \subseteq R$ be a set of primitive elements such that $R = R_0[S]$ for some A-algebra R. Then the* Witt construction *$(A, W_S(R))$ is defined by*

$$W_S(R) = R_0[S \times \mathbb{N}_0] = R_0[s_i \mid s \in S, i \in \mathbb{N}_0]$$

where $\Delta(s_i)$ is determined by requiring that the "ghost component"

$$p^i s_i + p^{i-1} s_{i-1}^p + \cdots + p s_1^{p^{i-1}} + s_0^{p^i}$$

be primitive. (Note: as usual, these elements are to be interpreted by using the universal formulas which they imply in the absence of $\mathbb{Z}$-torsion.)

5. **Lemma.** *The Hopf algebroid $(A, W_S(R))$, up to isomorphism, only depends on the images of the elements $s \in S$ in (47).*

Proof. The Witt construction is a pullback of a diagram of affine groupoid schemes

$$\begin{array}{ccc} (X,\Phi) & \xrightarrow{f} & (\bullet,\mathbb{G}_a) \\ & & \uparrow{\scriptstyle \pi} \\ & & (\bullet, G). \end{array} \tag{48}$$

Changing the representatives of the cohomology classes corresponds to choosing a morphism of affine schemes

$$w : X \to \mathbb{G}_a$$

and replacing f by g given on

$$\alpha : x \to y$$

by

$$g(\alpha) = f(\alpha) + w(x) - w(y)$$

where the addition denotes the operation in $\mathbb{G}_a$. Therefore, the conclusion of the Lemma holds if we can lift w to G:

$$\begin{array}{ccc} X & \xrightarrow{w} & \mathbb{G}_a \\ & \overset{\overline{w}}{\searrow} & \uparrow{\scriptstyle \pi} \\ & & G. \end{array}$$

The existence of lifting in our case follows from the fact that π is the *Spec* of the unit of a polynomial algebra. □

6. **Theorem.** *Let* $S = \{s_1, s_3, \dots\} \subset LS$ *be the elements represented by real projective spaces* $\mathbb{R}P^{4i+1}$, $i \in \mathbb{N}_0$. *Then, locally at* 2, *we have a canonical isomorphism of Hopf algebroids*

$$(L, LSC) \cong (L, W_S LS). \tag{49}$$

Proof. Since the Witt construction Hopf algebroid is commutative and generated by elements s where $\Delta(s)$ does not involve any elements of MU_* of dimensional degree > 0, it is given by a coaction of a Hopf algebra on a comodule algebra. Furthermore, this Hopf algebra is bipolynomial. Similar conclusions apply also to the Hopf algebroid (L, LSC) (see the proof of Theorem 3). Thus, we may apply Proposition 2.3 of Ravenel and Wilson [34]. □

Comment: It is worth noting now that our spectral sequences converging to $MO[2]$, MSC, despite being based on resolutions by MU-modules, are *not* the same as the Adams-Novikov spectral sequences

$s\backslash t-s$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	$\mathbb{Z}$		0		$\mathbb{Z}$		0		$\mathbb{Z}^2$		0		$\mathbb{Z}^3$
1		(4)		$\mathbb{Z}$		$(2,16)$		$\mathbb{Z}^2$		$(8,64)$		$\mathbb{Z}^4$	
2			0		(2)		(4)		$(2,4,8)$		$(\mathbb{Z},2,4)$		$(2,4^2,8,32)$
3				0		0		0		(2)		(2)	
4					0		0		0		0		0

Figure 1. Self-conjugate cobordism groups

for these spectra. A way to see this is to consider the generator $a_1 \in MSC_1 = MO[2]_1$, which corresponds to

$$(\overline{s}_1) \in Ext^1_{LSC}(MU_*, MU_*).$$

We see from (16) that applying the Hurewicz homomorpism

$$\pi_* MO[2] \to MU_* MSC \to H\mathbb{Z}/2_* MO[2], \tag{50}$$

the class a_1 goes to the class a_1. (This map is given, in fact, by taking the first Stiefel-Whitney class of the specified 1/2 of the stable normal bundle of $\mathbb{R}P^1$, which is the Möbius strip.) Thus, the class a_1 in the source of (50), which is 4-torsion, survives all the maps, and hence the middle term of (50) must have 4-torsion (the case of MSC is the same). We conclude that the Adams-Novikov cobar complexes for MSC, $MO[2]$ have torsion, while the rectified cobar complex does not.

Comment: The Witt construction can be described as a polynomial coalgebra on generators in topological degrees $2j$ for $j = 1, 2, \dots$. Thus, the E_2-term of the rectified Novikov spectral sequence for MSC can be described as (2).

On the other hand, there is a decreasing filtration on the Witt construction where for every primitive generator s, the iterated Verschiebung s_i is 2^i. This leads to an algebraic spectral sequence whose E_1-term is

$$Ext_A(\mathbb{Z}, \mathbb{Z}) \otimes MU_* \tag{51}$$

where the polynomial generators of A act on $\mathbb{Z}$ trivially (which is forced by dimensional degree). This leads to an analog of the May spectral sequence, which only has non-zero terms for

$$t - s \geq s^2$$

For calculations of MSC_*, see Figure 1 - the numbers indicate orders of cyclic summands; thus, for example, the group in dimension $t-s = 12$, $s = 2$ is $\mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/32$.

$s \setminus t-s$	0	1	2	3	4	5	6	7	8
0	$\mathbb{Z}_0$		0		$\mathbb{Z}_0$		0		$\mathbb{Z}_0^2$
1		$(4)_1$		$\mathbb{Z}_2$		$(4_1, 16_1)$		$(2_1, \mathbb{Z}_2, \mathbb{Z}_4)$	
2			0		(4_3)		(4_2)		$(4_3, 16_3, 4_5)$
3				0		0			

$s \setminus t-s$	9	10	11	12
0		0		$\mathbb{Z}_0^3$
1	$(2_1^2, 8_1, 64_1)$		$(2_1^2, \mathbb{Z}_2^2, \mathbb{Z}_4, \mathbb{Z}_8)$	
2		$(2_1, 4_1, 2_3, \mathbb{Z}_6)$		$(2_3^2, 8_3, 64_3, 4_5, 16_5, 4_3)$
3	(4_4)		(4_7)	

FIGURE 2. The algebraic rectified Adams-Novikov spectral sequence for MSC

For the E_2-term of the algebraic spectral sequence (51), see Figure 2. Subscripts of entries indicate their algebraic filtration degrees. We can see from the table that the algebraic spectral sequence has both higher differentials and extensions. For example, there is a d_2 from $(t-s, s) = (5, 1)$ to $(t-s, s) = (4, 2)$. There is an extension in $(t-s, s) = (7, 1)$.

4. THE COLLAPSE OF THE RECTIFIED ADAMS-NOVIKOV SPECTRAL SEQUENCE FOR MSC

In this section, we prove Theorem 1. We begin with a general observation. Let R be an E_∞ ring spectrum and let $\alpha_1, \dots, \alpha_n, \dots \in R_*$ be elements, and let M be an R-module. We are interested in the example

$$R = MSC,\ M = MU. \tag{52}$$

Then we can form an R-module

$$F_{(a_1,\dots,a_n,\dots)}(R) = \operatorname*{holim}_n \Sigma^{1-n} R/(\alpha_1, \dots, \alpha_n). \tag{53}$$

In fact, we can similarly form

$$F_{(a_1,\dots,a_n,\dots)}(M) = \operatorname*{holim}_n \Sigma^{1-n} M \wedge_R R/(\alpha_1, \dots, \alpha_n). \tag{54}$$

The comparison

$$M \wedge_R F_{(a_1,\dots,a_n,\dots)}(R) \xrightarrow{\sim} F_{(a_1,\dots,a_n,\dots)}(M) \tag{55}$$

is a matter of convergence, although the map always exists canonically and (55) holds in the case of (52). In our present setting, convergence holds due to increasing connectivity of maps between the fibers

$$\Sigma^{-n} R/(\alpha_1, \dots, \alpha_{n+1}) \to \Sigma^{1-n} R/(\alpha_1, \dots, \alpha_n).$$

Now note further that in the case (52), $F_{(a_1,\dots,a_n,\dots)}(M)$ maps into our cobar MSC-resolution with a map inducing an isomorphism on E_2-terms. (This simply follows from the fact that the elements a_i are permanent cycles, a known fact which is recalled in the Appendix.) Thus, we have

$$F_{(a_1,\dots,a_n,\dots)}(M) \sim R. \tag{56}$$

Together with (55), this implies, in fact, that $F_{(a_1,\dots,a_n,\dots)}(R)$ is a strong dual of M in the derived category DR of R-modules, and that in fact, more strongly, both objects are invertible and inverse to each other.

To see how this implies the collapse of our spectral sequence, we need to recall some more context. First of all, the algebraicity of (1) implies that there is a motivic version MSC^{Mot} of the spectrum MSC (we shall only work in the 2-complete motivic category over the field $\mathbb{C}$, suppressing the completion from the notation).

Additionally, by [15], Section 4, we have

$$MGL_\star = MU_*[\tau] \tag{57}$$

where the generators x_i, τ have dimensions $i(1+\alpha)$, $(1-\alpha)$ in the notation of [15], where the element τ was denoted θ. For general background on algebraic cobordism, we refer the reader to Morel and Levine [24]. The "$1,\alpha$" notation is motivated by analogs with $\mathbb{Z}/2$-equivariant homotopy theory via the Real realization - the analogy was noticed in the 1990's by Hu and Kriz, who used it in several subsequent papers. In recognition of the connections with algebraic geometry, it has become more common to denote the dimensions by $1=(1,0)$, $\alpha=(1,1)$.

Now our constructions may be repeated verbatim in the 2-completed motivic category over $\mathbb{C}$, we obtain a variant of the spectral sequence (2) of the form

$$Ext_A(\mathbb{Z}, MU_*)[\tau] \Rightarrow MSC^{Mot}_*. \tag{58}$$

Now we can use the result of Gheorghe [8] which asserts that when we change rings from S^{Mot} to S^{Mot}/τ, the resulting spectral sequence

$$Ext_A(\mathbb{Z}, MU_*) \Rightarrow (MSC^{Mot}/\tau)_* \tag{59}$$

collapses.

In fact, any higher differentials in (58) give rise to τ-torsion in MSC^{Mot}_*. Now let us return to the setup of the beginning of this section, this time putting

$$R = MSC^{Mot},\ M = MGL \tag{60}$$

$s\backslash\ t-s$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	$\mathbb{Z}$		0		$\mathbb{Z}$		0		$\mathbb{Z}^2$		0		$\mathbb{Z}^3$
1		(4)		0		$(2,16)$		0		$(8,64)$		0	
2			(2)		0		$(2,4)$		(2)		$(2,4^2)$		(4)
3				(2)		0		(2^2)		(2)		(2^5)	
4					(2)		0		(2^2)		(2)		(2^5)
5						(2)		0		(2^2)		(2)	
6							(2)		0		(2^2)		(2)
7								(2)		0		(2^2)	
8									(2)		0		(2^2)
9										(2)		0	
10											(2)		0
11												(2)	
12													(2)

FIGURE 3. The rectified Adams-Novikov spectral sequence for $MO[2]$

(still working in the 2-completed motivic category over $\mathbb{C}$). As above, we conclude again that $M = MGL$ is an invertible object in the derived category of R-modules. By (57), its homotopy groups have no τ-torsion. Suppose now $0 \neq b \in \pi_* R$ were τ-torsion. Therefore, the element b would have to act by 0 on M. However, since M is invertible, it would therefore also act by 0 on R, which is a contradiction.

The lack of $\mathbb{Z}$-multiplicative extensions is proved similarly: Suppose

$$2^m x = y\tau \tag{61}$$

for $x, y \in R_*$. Then the relation (61) will also be true in the corresponding operations on the invertible module M_*. However, in our case, $M_* = MU_*[\tau]$, whose operations are $MU^*MU[\tau]$ and thus, (61) does not occur.

Comment: We do not know if the spectral sequence

$$Ext_{LS}(L, L) \Rightarrow MO[2]_* \tag{62}$$

collapses to the E_2-term. However, recall that the primitive generators $\overline{s}_{2k+1}$ of degrees $(t-s, s) = (4k+1, 1)$ are permanent cycles (represented by $\mathbb{R}P^{4k+1}$). Now these manifolds all have non-zero Stiefel-Whitney numbers of the half-normal number. Equivalentnly, they produce a non-trivial image by the Hurewicz homomorphism into $H\mathbb{Z}/2_*(MO[2])$. Hence, all powers of these generators are non-zero (in contrast, for example, with Nishida's nilpotence theorem in the stable homotopy groups of spheres). See Figure 3.

5. Appendix: The classical methods

The subject of MSC was extensively studied, see for example [4, 6, 10, 23, 27, 28, 35]. We recall here some known partial results some of which are implicit in our discussion, and which are not easily quotable in the literature, at least in the present context.

Let us denote by $\mathcal{L}$ the subring of all elements $x \in MU_*$ such that

$$c_{2i+1} c_{j_1} \dots c_{j_\ell}[x] = 0$$

for all $i, j_1 \dots, j_\ell \in \mathbb{N}$.

7. **Theorem.** *The ring $\mathcal{L}$ coincides with the image of the canonical map $\iota : MSC_* \to MU_*$.*

To prove this, note that $Im(\iota) \subseteq \mathcal{L}$ was proved by Buchstaber [4], Lemma 24.17. We also have

5.1. **Proposition.** *(Buchstaber [4], Theorem 24.20) If we denote by $\kappa : MU_* \to MO_*$ the canonical map, then*

$$Im(\kappa\iota) = \kappa(\mathcal{L}). \tag{63}$$

□

5.2. **Proposition.** *Let*

$$\beta_n = b_n^2 - 2b_{n-1}b_{n+1} + \dots + 2(-1)^n b_{2n} \in MU_* \otimes \mathbb{Q}$$

where

$$b(x) = x + \sum_{n \geq 1} b_n x^{n+1}$$

is the exponential series of the universal formal group law. Then

$$\mathcal{L} \otimes \mathbb{Q} = \mathbb{Q}[\beta_1, \beta_2, \dots]. \tag{64}$$

Proof. The series

$$\beta(x) = -x^2 + \sum_{n \geq 1} \beta_n (-x^2)^{n+1}$$

satisfies

$$\beta(x) = b(x)b(-x) = b(x)ib(x)$$

where $i(x)$ is the formal inverse. Thus, considering

$$b : \mathbb{C}P^\infty \to \Sigma^2 MU_{\mathbb{Q}},\ \beta : \mathbb{C}P^\infty \to \Sigma^4 MU_{\mathbb{Q}},$$

we can write

$$\beta = b\overline{b}$$

where $\overline{b}$ denotes complex conjugation. In other words, β can be expressed as the composition

$$\mathbb{C}P^\infty \xrightarrow{\phi} BU(2) \xrightarrow{c_2} \Sigma^4 MU_{\mathbb{Q}}$$

where c_2 is the Conner-Floyd Chern class and ϕ is B applied to the embedding

$$S^1 \to U(2)$$

by

$$z \mapsto \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}.$$

Thus, the map κ factors as

$$\mathbb{C}P^\infty \to BO(2) \to BU(2)$$

where the second map is complexification. Therefore, $\beta_n \in \mathcal{L} \otimes \mathbb{Q}$ by the fact that rationally, odd Chern classes vanish on a complexified real bundle. On the other hand, it follows from considering rational homology that $Im(\iota) \otimes \mathbb{Q}$ is a polynomial algebra on generators in dimensions divisible by 4 (for example by Conner-Floyd [6]), and thus our statement follows from a counting argument. □

Recall the Milnor class $s_n = p_n(c_1, c_2, \dots, c_n)$ in Chern classes where

$$p_n(\sigma_1, \sigma_2, \dots) = t_1^n + t_2^n + \dots$$

where σ_i are the elementary symmetric polynomials in the t_i. Recall that the Milnor number $s_n[x]$ detecs the image of an element $x \in MU_{2n}$ in the module of indecomposables QMU_{2n}.

5.3. **Proposition.** *There exists an element $V_k \in Im(\iota)_{2(2^k-1)}$ whose Milnor number is 8. These elements are equal to v_k^2 (where we denote $v_k = x_{2^k-1}$) modulo other monomials in the x_i. Addionally, V_k can be chosen so that $\kappa(V_k) = 0$.*

Proof. For the first statement, it suffices to construct an element in the given dimension with Milnor number of 2-valuation 3 (since at odd primes, MSC is just a wedge of copies of BP, see [6]). Now for $k \geq 3$, one notes that

$$v_2 \binom{2^{k+1} + 2^k - 4}{2^{k+1} - 7} = 3.$$

It follows that in this case, we can take a Stong manifold given as the $\mathbb{Z}/2$-quotient of an intersection of $2^k - 2$ hypersurfaces of bidegree $(1,1)$ in

$$\mathbb{C}P^{2^{k+1}-7} \times \mathbb{C}P^{2^k+3}$$

by the diagonal $\mathbb{Z}/2$-involution on both $\mathbb{C}P^{2i+1}$-factors:

$$(z_0, z_1, \dots, z_{2i}, z_{2i+1}) \mapsto (-\overline{z_1}, \overline{z_0}, \dots, -\overline{z_{2i+1}}, \overline{z_{2i}}).$$

For $k = 1, 2$, one must use other generators (e.g. [27] observes that the statement of Theorem 7 is true in dimensions ≤ 128).

Now by Proposition 5.2, V_k is congruent to $4\beta_{2^k-1}$ modulo the square of the augmentation ideal in $Im(\iota) \otimes \mathbb{Q}$. We see that no multiples of the monomials of $\beta_i\beta_j$ contain v_k^2, which is a monomial summand of $4\beta_{2^k-1}$. The second statement follows. Finally, for the last statement, by [28, 4], $Im(\kappa\iota)$ is the 4th power of the Floyd ring. In particular, it is a polynomial ring with generators in dimensions $8(2i+1)$, $8 \cdot 2^\ell$, $8 \cdot i$ where i is not a power of 2. Therefore, lifting the generators to $Im(\iota)$, none of them can contain a rational multiple of β_{2^k-1} as a summand (for reasons of dimension). Therefore, adding a polynomial in these generators cannot cancel the term v_k^2. □

Proof of Theorem 7. It remains to prove that

$$\mathcal{L} \subseteq Im(\iota). \tag{65}$$

Since (as we already noted) the problem is trivial at odd primes, we may work completed at 2. Thus, suppose $y \in \mathcal{L}_2^\wedge$. By Proposition 5.2, we may write

$$y = p(V_1, V_2, \dots)$$

where p is a polynomial with coefficients in

$$\mathbb{Q}_2[x_i \mid i \neq 2^k - 1].$$

However, the coefficient a_ℓ where $\ell = (\ell_1, \ell_2, \dots)$ of $V_1^{\ell_1} V_2^{\ell_2} \dots$ must in fact satisfy

$$a_\ell \in \mathbb{Z}_2[x_i \mid i \neq 2^k - 1],$$

since otherwise the element y would not belong to $(MU_*)_2^\wedge$ (consider the coefficient of $v_1^{2\ell_1} v_2^{2\ell_2} \dots$). Additionally, we must also have $a_\ell \in \mathbb{Q}_2[\beta_1, \beta_2, \dots]$ (since the element of highest degree which fails this condition would contradict Proposition 5.2).

Now by Proposition 5.1, there exist $b_\ell \in Im(\iota)$ so that

$$b_\ell - a_\ell \in (2, v_1, v_2, \dots).$$

Now since we also have $b_\ell - a_\ell \in \mathcal{L}$, it will have a lower degree (and hence be subject to induction) except when $\ell = 0$. However, the constant term of $b_0 - a_0$ is now divisible by 2. Thus, we may divide the constant term by 2 and apply the same procedure to it, and apply induction to its other coefficients. Since we are working completed at 2, the infinite sum in increasing powers of 2 we produce by repeating this process will converge to an element of $Im(\iota)^{\wedge}_2$ which is equal to y.

□

Suppose now we filter LSC by 1/2 times the topological degree of the L-degree of the augmentation $LSC \to L$. This is a decreasing filtration, and one has an algebraic Novikov spectral sequence

$$E_2 = Cotor_{(L,E_0 LSC)}(L,L) \Rightarrow Cotor_{(L,LSC)}(L,L). \tag{66}$$

Moreover, it follows from the discussion of the previous section that the left hand side of (66) is of the form

$$Cotor_{(L,E_0 LSC)}(L,L) = \Lambda_L(a_1, a_3, a_5, \dots) \tag{67}$$

where the generator a_{2k+1} is in degree $2k+1$. Moreover, it follows from considering the Adams spectral sequence that the generators a_{4k+1} are realized by the manifolds $\mathbb{R}P^{4k+1}$ whose stable tangent bundle is $(4k+2)\gamma^1_{\mathbb{R}}$, which is double a real bundle, and thus canonically has a structure of a self-conjugate complex bundle.

On the other hand, the representatives N^{4k-1} of a_{4k-1} were constructed by Landweber [23, 35]. They are given by

$$S^{4k-1} \times_{Sp(1)} S^3$$

where $Sp(1)$ acts on S^{4k-1} (thought of as the unit sphere in $\mathbb{H}^k$) by right multiplication of quaternions, and on $S^3 = Sp(1)$ by conjugation. (Here we are considering only the compact form of $Sp(k)$.)

One then remarks that the sum of the tangent bundle of N^{4k-1} and a 1-dimensional trivial real bundle is isomorphic to $k\gamma^1_{\mathbb{H}}$, and thus has a canonical structure of a self-conjugate complex bundle.

5.4. **Proposition.** *The Toda brackets*

$$\langle a_{2k+1}, a_{2k+1}, \dots, a_{2k+1} \rangle \tag{68}$$

all contain $0 \in MSC_*$.

Proof. We begin by showing that

$$a^2_{2k+1} = 0. \tag{69}$$

When k is even, consider the manifold with boundary

$$M = \mathbb{R}P^{4k+1} \times \mathbb{R}P^{4k+1} \times [0,1]$$

with $\mathbb{Z}/2$-action by

$$(x, y, t) \mapsto (y, x, 1-t). \tag{70}$$

The fixed point submanifold is

$$\Delta \times \{1/2\}$$

where $\Delta \subset \mathbb{R}P^{4k+1} \times \mathbb{R}P^{4k+1}$ is the diagonal. Thus, the normal bundle

$$\nu_\Delta^M = (4k+2)\gamma_{\mathbb{R}}^1,$$

with $\mathbb{Z}/2$-action by

$$x \mapsto -x. \tag{71}$$

This is canonically a complex bundle, so we can perform a complex blow-up of Δ in N and form a non-singular manifold Z by taking the $\mathbb{Z}/2$-quotient (since the submanifold of $\mathbb{Z}/2$-fixed points now has complex codimension 1). Moreover, the construction just performed is complex and self-conjugate, thus proving that the manifold Z with boundary has an MSC-structure, thus providing the cobordism which proves (69).

In the case of k odd, we put, analogously,

$$M = N^{4k-1} \times N^{4k-1} \times [0,1],$$

again with $\mathbb{Z}/2$-action by (70). This time, the fixed point manifold is

$$E \times \{1/2\}$$

where $E \subset N^{4k-1} \times N^{4k-1}$ is the diagonal. Thus, the normal bundle is

$$\nu_E^M = k\gamma_{\mathbb{H}}^1.$$

Once again, this is naturally a complex bundle, so we can perform a complex blow-up of E, and then take a $\mathbb{Z}/2$-quotient, thus again getting a manifold with boundary Z. Since, again, the construction performed is complex and self-conjugate, Z is an MSC-cobordism, again proving (69).

Now assume an MSC-cobordism Z_n is is constructed proving (67) with n factors. If k is even, we form a manifold M_n by gluing $Z_n \times \mathbb{R}P^{4k+1}$ and $\mathbb{R}P^{4k+1}$ along

$$\mathbb{R}P^{4k+1} \times M_{n-1} \times \mathbb{R}P^{4k+1}$$

and multiplying by $[0,1]$. The manifold M_n has a natural action by reversing the order of the copies of $\mathbb{R}P^{4k+1}$ (and extending by the corresponding maps on the cobordism coordinates), and mapping

$$t \mapsto 1-t$$

on the new interval coordinate. The action is free on the previous cobordism coordinates, so the fixed point manifold D_n is $\{1/2\}$ times the fixed point of the $\mathbb{Z}/2$-action on

$$(\mathbb{R}P^{4k+1})^n$$

by reversing the order of factors. This is a diagonal manifold isomorphic to $(\mathbb{R}P^{4k+1})^{\lceil n/2\rceil}$, and its normal bundle is a sum of $\lfloor n/2\rfloor$ copies of the $(4k+2)\gamma^1_{\mathbb{R}}$ on the individual coordinates, and a trivial bundle, with $\mathbb{Z}/2$-action by (71). (Because of the previous cobordism coordinates, there are always enough trivial coordinates to stabilize.) Thus, the normal bundle of D_n in M_n is a self-conjugate complex bundle, and we can again perform a complex blow-up of D_n in M_n, and then take a $\mathbb{Z}/2$-quotient. The construction is complex and self-conjugate, and thus, we obtain the required cobordism proving (67) with $n+1$ factors.

The case of k odd is completely analogous, with $\mathbb{R}P^{4k+1}$ replaced by N^{4k-1}. □

References

[1] S.Araki: τ-cohomology theories, *Japan Journal of Mathematics* (N.S.) 4 (2) (1978) 363-416, Forgetful spectral sequences, *Osaka Journal of Mathematics* 16 (1) (1979) 173-199, Orientations in q-cohomology theories, *Japan Journal of Mathematics* (N.S.) 5 (2) (1979) 403-430.

[2] M.F.Atiyah: *K -theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1967

[3] M.F.Atiyah: K-theory and reality, *Quart. J. Math. Oxford Ser.* (2) 17 (1966), 367-386

[4] V. M. Buchstaber: Characteristic classes in cobordism and topological applications of the theory of single-valued and two-valued formal groups, *Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat.* Volume 10, (1978) 5-178

[5] V.M.Buchstaber, S.P.Novikov: Formal groups, power systems and Adams operators, *Mat. Sb.* (N.S.) 84 (126) (1971), 81-118

[6] P. E. Conner, E. E. Floyd: Torsion in SU-bordism, *Mem. Amer. Math. Soc.* 60 (1966)

[7] A.Elmendorf, I.Kriz, M.Mandell and J.P.May: *Rings, Modules and Algebras in Stable Homotopy Theory*, AMS Surveys and Monographs series 47, American Mathematical Society, 1997

[8] B. Gheorghe: The Motivic Cofiber of τ, *Doc. Math.* 23 (2018), 1077–1127

[9] R. Godement: Topologie Algèbrique et Théorie des Faisceaux, *Actualités Sci. Ind.* No. 1252. Publ. Math. Univ. Strasbourg. No. 13 Hermann, Paris 1958

[10] N. Y. Gozman: On the image of the self-conjugate cobordism ring in the complex and unoriented cobordism rings, *Dokl. Akad. Nauk SSSR*, Volume 216, Number 6, (1974), 1212-1214

[11] V.K.A.M.Gugenheim, J.P. May: *On the theory and applications of differential torsion products*, Memoirs of the American Mathematical Society, No. 142. American Mathematical Society, Providence, R.I., 1974

[12] M.A.Hill, M.J.Hopkins, D.C.Ravenel: On the nonexistence of elements of Kervaire invariant one, *Ann. of Math.* (2) 184 (2016), no. 1, 1-262

[13] P.Hu: The cobordism of Real manifolds, *Fund. Math.* 161 (1999), no. 1-2, 119-136

[14] P.Hu and I.Kriz: Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence, *Topology* 40 (2001), no. 2, 317-399

[15] P.Hu, I.Kriz, K.Ormsby: Remarks on motivic homotopy theory over algebraically closed fields, *J. K-Theory* 7 (2011), no. 1, 55-89

[16] P. Hu, I. Kriz, and K. Ormsby: The homotopy limit problem for Hermitian K-theory, equivariant motivic homotopy theory and motivic Real cobordism, *Advances in Mathematics* 228 (2011), no. 1, 434–480

[17] D. Husemoller: The structure of the Hopf algebra $H^*(BU)$ over a $\mathbb{Z}_{(p)}$-algebra, *Am. J. of Math.* 43, (1971), 329–349

[18] D.C.Isaksen, G.Wang, Z.Xu: Stable homotopy groups of spheres: from dimension 0 to 90.(English summary), *Publ. Math. Inst. Hautes Études Sci.* 137(2023), 107-243

[19] N.Kitchloo, W.S.Wilson: The $ER(n)$-cohomology of $BO(q)$ and real Johnson-Wilson orientations for vector bundles, *Bulletin of the London Math. Soc.*, 47 no. 5 (2015), 835-847

[20] A.Kono, M.Nagata: Homology of $(KSC)_n$ over the Dyer-Lashof algebra, *Japan. J. Math.* 6 (1980) 45-60

[21] P.S. Landweber: Conjugations on complex manifolds and equivariant homotopy of MU_*, *Bulletin of the American Mathematical Society* 74 (1968) 271-274

[22] P.S. Landweber: Cobordism operations and Hopf algebras, *Trans. Amer. Math. Soc.* 129 (1967), 94-110

[23] P. S. Landweber: On the symplectic bordism groups of the spaces $Sp(n)$, $HP(n)$ and $BSp(n)$, *Michigan Math.* J. 15 (1968), 145–153

[24] M. Levine, F. Morel: Cobordisme algébrique I, *C. R. Acad. Sci. Paris Sér. I Math.* 332 (2001), no. 8, 723-728

[25] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure: *Equivariant stable homotopy theory*, Vol 1213 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1986.

[26] J.W.Milnor: On the cobordism ring Ω_* and a complex analogue, *Amer. J. Math.* 82 (1960), 505-521.

[27] R. Nadiradze: Characteristic classes in SC-theory and their applications, *Proceedings of A. Razmadze Mathematical Institute*, Volume 104, (1995) 55-74

[28] R. Nadiradze: Involutions on Stong manifolds and their applications in cobordism theory, *Russ. Math. Surv.* 35 (1980) 264

[29] S.P.Novikov: Some problems in the topology of manifolds connected with the theory of Thom spaces, *Dokl. Akad. Nauk. SSSR.* 132 (1960), 1031-1034

[30] S.P.Novikov: The methods of algebraic topology from the viewpoint of cobordism theories, *Izv. Akad. Nauk. SSSR. Ser. Mat.* 31 (1967), 855-951

[31] S.P.Novikov: Homotopy properties of Thom complexes, *Mat. Sb.* (N.S.) 57 (99) (1962), 407-442

[32] S. Priddy: Dyer-Lashof operations for the classifying spaces of certain matrix groups, *Quart. J. Math.* Oxford Ser. (2), 26(102) (1975) 179–193

[33] D. C. Ravenel: *Complex cobordism and stable homotopy groups of spheres*, AMS Chelsea Publishing, 2003

[34] D.C.Ravenel, W.S.Wilson: Bipolynomial Hopf Algebras, *Journal P. Applied Alg.* 4 (1974) 41-45

[35] L.Smith, R.E.Stong: The structure of BSC, *Inventionnes Math.* 5 (1968) 138-159

[36] R. Thom: Quelques propriétés globales des varietés differentiables, *Comm. Math. Helv.* 28 (1954), 109-181.