WHAT IS AN EQUIVARIANT ADAMS SPECTRAL SEQUENCE?

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Abstract. While the Adams spectral sequence is a fundamental tool for computing non-equivariant stable homotopy groups from ordinary homology groups, no analogous universally applicable tool is known equivariantly. This is due in part to the variety of inequivalent generalizations of ordinary homology in the equivariant setting, and in part to the added complexity of each such theory. In this note, we exhibit an equivariant Adams spectral sequence, where the role of 'ordinary' homology is played by a collection of spectra represented by chain complexes of Mackey functors for subgroups of G. In the process, we revisit concepts introduced by Guillou–May, Kaledin, Barwick, and others in a new light.

1. Introduction

Similarly as non-equivariantly, for a finite group G-equivariant ordinary homology theory is defined as a (genuine, i.e. $RO(G)$-graded) G-equivariant spectrum whose $\mathbb{Z}$-graded Mackey-functor valued homotopy groups are concentrated in degree 0. (For a review of these concepts, we recommend [12, 13].) Mackey functors form a tensor (abelian) category, where the unit of the tensor product, denoted by $\square$, i the Burnside Mackey functor $\mathcal{A} = \mathcal{A}_G$ (see e.g. [4]). Accordingly, $\mathcal{A}_G$ can be considered universal coefficients for $RO(G)$-graded equivariant (co)homology. Non-equivariantly, of course, the Eilenberg-Mac Lane spectrum with universal coefficients $H|Z$ is not a good basis for the Adams spectral sequence, since $H\mathbb{Z}$ is not a flat spectrum in the sense that $H\mathbb{Z}_*H\mathbb{Z}$ is not flat over $H\mathbb{Z}_*$. One uses the flat spectrum $H\mathbb{Z}/p$ instead, which, for bounded below spectra of finite type (i.e. cell spectra with cells of dimensions bounded below, and finitely many cells in each dimension), gives the familiar Adams spectral sequence converging to $\pi_*X_p^\wedge$.

Equivariantly, however, $H\mathcal{A}/p$ cannot be expected to be a flat spectrum. One might expect that a better candidate may be the constant Mackey functor spectrum $H\underline{\mathbb{Z}/p}$. This has in fact been shown to work in [9] for $p = 2$ and $G = \mathbb{Z}/2$, when we consider the $RO(G)$-graded

Steenrod algebra (and converging to $RO(G)$-graded homotopy groups). However, this already becomes much more problematic for $G = \mathbb{Z}/p$. Sankar and Wilson [15] showed that for $G = \mathbb{Z}/p$, $Hunderline\mathbb{Z}/p$ is not a flat spectrum, even in the $RO(G)$-graded sense. This can still be partially remedied. In [6], the $RO(\mathbb{Z}/p)$-graded $H\underline{\mathbb{Z}/p}$-based dual Steenrod algebra has been fully calculated. Flatness can be partially restored (modulo free spectra in a suitable sense) if along with the Mackey functor $\underline{\mathbb{Z}/p}$, one includes the Mackey functor $\widetilde{\mathcal{L}}_{p-1}$ which is 0 on the fixed orbit and the integral reduced regular $\mathbb{Z}/p$-representation on the free orbit (see [8]). In any case, this approach does not seem workable for larger groups G.

Kaledin [10] investigated *derived Mackey functors*, which are essentially equivalent to $i_\sharp(H\mathbb{Z})$-modules, where $i_\sharp$ denotes the pushforward from non-equivariant to G-equivariant spectra indexed on a complete universe. The point is that for a commutative ring R, $i_\sharp HR$ is an E_∞-ring spectrum, and therefore has a nice derived category of E_∞-modules, which is equivalent to the derived Mackey functors of [10] (also related to the concepts of [1]).

Kaledin remarked that derived Mackey functors behave, in many respects, better than Mackey-valued G-equivariant ordinary (co)homology. For example, for any finite group G, $i_\sharp H\mathbb{Z}/p$ is always a flat spectrum and the Adams spectral sequence based on it always converges to the p-completed homotopy groups of a bounded below G-spectrum X of finite type. This is simply because we have

$$\pi_*(i_\sharp HR \wedge X) = HR_*(X^G),$$

in particular

$$\pi_*(i_\sharp H\mathbb{Z}/p \wedge i_\sharp H\mathbb{Z}/p) = \pi_* H\mathbb{Z}/p \otimes A_*$$

where A_* denotes the non-equivariant dual Steenrod algebra, so the $i_\sharp H\mathbb{Z}/p$-based Adams spectral sequence is simply

$$Cotor_{A_*}(Z/p, H\mathbb{Z}/p_*(X^G)) \Rightarrow \pi_* X^\wedge_p., \tag{1}$$

i.e. the non-equivariant Adams spectral sequence for X^G. It may also be helpful to recall that by the Segal-tom Dieck splitting,

$$\pi_*(i_\sharp HR) = \bigoplus_{(H)\subseteq G} HR_*(BW(H)).$$

It is worth noting that there are other ways in which derived Mackey functors behave well. For example, they have a spectral version, which can be used to characterize the derived category of G-spectra (i.e. the G-equivariant stable homotopy category), [2, 5].

The drawback of using $i_\sharp H\mathbb{Z}/p$ as a base of an Adams spectral sequence as in (1) is that, in some sense, we have not gained anything: if we knew the homology of the fixed point spectrum X^G, there is no need of an equivariant Adams spectral sequence. Maybe more to the point, derived Mackey funcrors do not really have a model that would allow, say, an algorithmic chain-level computation of $(i_\sharp HR)_*X$, for, say, a finite G-CW-complex X.

This brings up the main point of the present note. For us, an ordinary (co)homology theory means a G-equivariant spectrum which is realizable by a chain complex of Mackey functors. We will prove the following

1. **Theorem.** *For a finite group G, there exists a diagram $\mathcal{D}_G$ of E_∞-G-equivariant ring spectra (in fact realizable by E_∞-algebras in chain complexes of G-Mackey functors and E_∞-morphisms such that*

$$i_\sharp H\mathbb{Z} = \operatorname{holim} \mathcal{D}_G \tag{2}$$

as an E_∞-ring spectrum. A similar statement also holds with $\mathbb{Z}$ replaced by any commutative ring.

The precise statement will be made in the next section. However, recall that for a projection $\psi_H : G \to G/H$, and a G/H-Mackey functor M, we have a G-Mackey functor $\psi_H^* M$, whose value on isotropy J is the value of M on isotropy J/H when $J \supseteq H$, and 0 otherwise.R ecall also that For $H \subseteq G$, there exists a G-representation α_H such that $\alpha_H^J = 0$ if and only if $J \supseteq H$. In those terms, the entries of the diagram can be written as

$$H(\psi_K^* \mathcal{A}_K) \wedge S^{\infty\alpha_H}, \quad K \subseteq H.$$

Denote, for any G-representation V,

$$a_V : S^0 \to S^V$$

the inclusion. Then, Theorem 1 has a corollary for computing the homology of X^G for a G-spectrum X:

2. **Corollary.** *For a G-spectrum X, there exists a spectral sequence*

$$R^s \lim_{\leftarrow} (H(\psi_K^* \mathcal{A}_K)_\star(X)[a_{\alpha_H}^{-1}]_t) \Rightarrow H\mathbb{Z}_{t-s}(X^G). \tag{3}$$

Here the subscript $?_\star$ denotes $RO(G)$-graded coefficients, as in [9]. (We could, alternatively, write the homology group on the ordinary homology on the left hand side of (3) as

$$(\psi_K^* \mathcal{A}_K \otimes_{\mathcal{O}_G^{Op}} \widetilde{C}_*(S^{\infty\alpha_H}))_t(X) \tag{4}$$

Note that this does count as "ordinary" equivariant homology in the sense we defined it. Again, the precise form of the diagram will be described in the next section.

One should note that (2) cannot really be considered a chain-level construction involving Mackey functors. If it were, then $i_\sharp H\mathbb{Z}$ would be realized by a chain complex of Mackey functors, i.e. an ordinary G-equivariant (co)homology in our sense, which would make it an $H\mathcal{A}_G$-module. This is not the case (as will be seen in the next Section). Thus, a spectral sequence albeit "short," is the best we can do.

We will see that an analogous statement to Corollary 2 holds with coefficients in any abelian group. Then one interesting feature of (3) is that it specifies a decreasing filtration on the ordinary homology of X^G for a G-spectrum X. We call this the *Hodge filtration*. It is important to note that this filtration cannot exist spectrally. While the sphere spectrum can be considered as a spectral analogue of the derived Mackey functor corresponding to $i_\sharp(H\mathbb{Z})$ via the results of [2, 5], there is no spectral analogue of $H\mathcal{A}_G$. We will return to this point in Section 3.

Accordingly, the Hodge filtration does not preserve Steenrod action, but exhibits a more delicate behavior instead. For illustration, we will work out the example of $RO(G)$-graded spheres for $G = \mathbb{Z}/p$.

More generally, for primary cyclic groups, it turns out that the non-trivial information in the Hodge filtration is $F^0 \subseteq F^1$. We shall also work that out explicitly.

In this context, a couple of peculiar properties of ordinary equivariant (co)homology should be mentioned. One is that $\widetilde{C}_*(S^{\infty\alpha_H})$, while obviouly an E_∞-algebra in chains, cannot actually be represented by a strictly graded-commutative DGA. Thus, the building blocks of $i_\sharp H\mathbb{Z}$, while still representing "ordinary" (o)homology theories, are actually not strictly commutative in the same way as Green functors, and only have E_∞-commutativity. As already mentioned, the diagram $\mathcal{D}$ cannot be realized on chain level, even though its individual arrows can be. This is due to a curious "tilting" ot ech entry of the diagrm, whereby an equivalent E_∞ ring spectrum has two different chain-level realizations which cannot be ralated no chain level.

Another strange effect is that the forgetful functor from the derived category of G-Mackey functors to G-spectra is not faithful. An example actually is given by the first k-invariant of $\widetilde{C}_*(S^{\infty\alpha_H})$. We will discuss these points in Section 3 below.

2. A model for $i_\sharp H\mathbb{Z}$ and related spectral sequences

2.1. **The abelian case.** Our construction is easier to describe in the case of an abelian group. For that reason, we treat this case first. Thus, let G be a finite abelian group. Consider the partially ordered set $\mathcal{P} = \mathcal{P}_G$ whose elements are pairs of subgroups $K \subseteq H$, and we put

$$(K_1 \subseteq H_1) \leq (K_2 \subseteq H_2) \tag{5}$$

when

$$H_1 \subseteq H_2, \quad K_1 \supseteq K_2.$$

We will exhibit a functor

$$\mathcal{D} = \mathcal{D}_G : \mathcal{P} \to G\text{-}E_\infty\text{-ring spectra}.$$

To this end, for a subgroup $H \subseteq G$, we recall the representation α_H where for a subgroup $J \subseteq G$, $\alpha_H^J = 0$ if and only if $H \subseteq J$. Then the suspension spectrum of $S^{\infty\alpha_H}$ is an E_∞-ring spectrum which can be used as a model for the unreduced suspension $\widetilde{E\mathcal{F}[H]}$, where $\mathcal{F}[H]$ is the family of subgroups not containing H and for a family $\mathcal{F}$, $E\mathcal{F}$ is a G-CW-complex whose J-fixed points are contractible for $J \in \mathcal{F}$, and empty otherwise.

Now, for a subgroup $H \subseteq G$, consider the functor

$$\phi_H : S^{\infty\alpha_H}\text{-Modules} \to G/H\text{-Spectra} \tag{6}$$

(where by modules, we mean E_∞-modules), given by

$$X \mapsto X^H.$$

It is known (see, for example , [7]) that ϕ is a right adjoint, and induces an equivalence on derived categories. We also note that the fixed point functor preserves E_∞-ring spectra, and that we have an E_∞-representative

$$\phi_H^{-1} H\mathcal{A}_H = H\psi_H^* \mathcal{A}_{G/H} \tag{7}$$

where for a G/H-Mackey functor M, $\psi_H^*(M)$ is the Mackey functor which on the orbit G/K is equal to $M(G/K)$, and is 0 on other orbits. The counit of adjunction then gives an E_∞-morphism

$$H\psi_H^*\mathcal{A}_{G/H} \to H\mathcal{A}_G \wedge S^{\infty\alpha_H} \tag{8}$$

In fact, applying the same principle, we get an E_∞-morphism

$$H\psi_{K_1}^*\mathcal{A}_{G/K_1} \wedge S^{\infty\alpha_{H_1}} \to H\psi_{K_2}^*\mathcal{A}_{G/K_2} \wedge S^{\infty\alpha_{H_2}} \tag{9}$$

when (5) holds (since we can replace G by G/K_2 and assume $\alpha_{H_1} \subseteq \alpha_{H_2}$). This gives a functor

$$\mathcal{D} = \mathcal{D}_G : \mathcal{P}_G \to G\text{-}E_\infty\text{-ring spectra}. \tag{10}$$

We can now restate Theorem 1 more precisely as

3. **Theorem.** *There is a canonical morphism of G-E_∞-ring spectra*

$$\iota_G : i_\sharp H\mathbb{Z} \to \underset{\leftarrow}{\operatorname{holim}}\, \mathcal{D}_G \tag{11}$$

which is an equivalence.

Proof. The morphism ι_G of (11) is given by the fact that $i_\sharp$ is left adjoint to G-fixed points. To prove that this morphism is an equivalence, we will use the fact that a morphism f of G-spectra is an equivalence if and only if

$$\Phi^J(f) = (f \wedge S^{\infty\alpha_J})^J \tag{12}$$

is a non-equivariant equivalence for every subgroup $J \subseteq G$.

To verify this property for ι_G, we apply Φ^J to every entry of the diagram $\mathcal{D}_G$. First note that when doing so, all $(K \subseteq H)$-terms become 0 unless

$$H \subseteq J. \tag{13}$$

Next note that all arrows (5) of the diagram $\Phi^J\mathcal{D}_G$ where (13) holds (i.e. where $H_2 \subseteq J$) and where $K_1 = K_2$ are equivalences by definition. This means that for the purposes of calculating Φ^J, we only need to consider the part of the diagram $\Phi^J\mathcal{D}_G$ on entries of the form $(K \subseteq J)$.

However, by the definition of a homotopy limit, it then suffices to restrict attention to the term $(J \subseteq J)$, i.e. $\Phi^J H\mathcal{A}_{G/J}$, which is $H\mathbb{Z}$ non-equivariantly. This completes the argument.

□

Remark: One may apply $? \otimes_{\mathbb{Z}} R$ for a commutative ring R to all the constructions up to this points and all the statements and arguments therefore hold with coefficients in R. In fact, R can be any abelian group, although in that generality, the diagram is only in the category of spectra (not E_∞-ring spectra).

2.2. **The case of $\mathbb{Z}/p$.** Let us consider the case of $G = \mathbb{Z}/p$. (For simplicity, we limit the discussion to $R = \mathbb{Z}$.) Let β be a faithful one-dimensional complex representation of $\mathbb{Z}/p$. Then the diagram $\mathcal{D}_{\mathbb{Z}/p}$ becomes

$$\begin{array}{ccc} & & H\mathbb{Z}^{\phi} \\ & & \downarrow \\ H\mathcal{A}_{\mathbb{Z}/p} & \longrightarrow & H(\mathcal{A} \otimes_{\mathcal{O}_{\mathbb{Z}/p}^{Op}} \widetilde{C}_*(S^{\infty\beta})) \end{array} \tag{14}$$

where $\mathbb{Z}^{\phi}$ is the $\mathbb{Z}/p$-Mackey functor with value $\mathbb{Z}$ in isotropy $\mathbb{Z}/p$ and value 0 in isotropy $\{0\}$.

Now the $RO(\mathbb{Z}/p)$-graded coefficients $(H\mathcal{A}_{\mathbb{Z}/p})_\star$ were calculated by Stong [16]. For simplicity, let us localize at p, where the choice of β does not matter, as p-local $\mathbb{Z}/p$-equivariant homology is periodic in a difference of choices of β, cf. [11]. Stong's calculation can be easily recoverd by considering the Burnside Mackey functor valued homology and cohomology of the CW-complex $S^{\ell\beta}$. For $p > 2$, the behavior of

$$(H\mathcal{A}_{\mathbb{Z}/p})_{k+\ell\beta} \tag{15}$$

depends on k. For $k = 0$, (15) is $\mathcal{A}_{\mathbb{Z}/p} = \mathbb{Z} \oplus \mathbb{Z}$ for $\ell = 0$, the augmentation ideal $I \cong \mathbb{Z}$ for $\ell > 0$ and the quotient $\mathcal{A}/I \cong \mathbb{Z}$ for $\ell < 0$.

For $k > 0$, (15) behaves like Borel cohomology. This means that we get $\mathcal{A}/I \cong \mathbb{Z}$ for k even and $\ell = -k/2$, and $\mathbb{Z}/p$ for $\ell < -k/2$, and 0 otherwise.

For $k < 0$, (15) behaves like Borel homology. This means that we get $I \cong \mathbb{Z}$ for $k < 0$ even and $\ell = -k/2$, $\mathbb{Z}/p$ for $k < -1$ odd and $\ell \geq -(1+k)/2$, and 0 otherwise.

This accounts for the lower left corner of the diagram (14). The rest of the diagram is β-periodic. The lower right corner can be identified with the Mackey-valued chain complex of the usual periodic $\mathbb{Z}/p$-CW-structure on $S^{\infty\beta}$. Denoting, as in [8], by $\underline{\mathcal{L}}_p$ the free $\mathbb{Z}/p$-Mackey functor on one generator in isotropy $\{0\}$ (which is $\mathbb{Z}$ in isotropy $\mathbb{Z}/p$ and $\mathbb{Z}[\mathbb{Z}/p]$ in isotropy $\{0\}$), this Mackey functor-valued chain complex becomes

$$\mathcal{A} \xleftarrow{1} \underline{\mathcal{L}}_p \xleftarrow{1-\gamma} \underline{\mathcal{L}}_p \xleftarrow{1+\gamma+\cdots+\gamma^{p-1}} \underline{\mathcal{L}}_p \longleftarrow \dots \tag{16}$$

(where γ denotes the generator of $\mathbb{Z}/p$ and the arrows are labelled by the image of the free generator).

We will study the chain complex (16) further in the next section. For the moment, let us confine ourselves to observing that its chain homology is

$$\mathbb{Z}^{\phi} \qquad 0 \qquad \mathbb{Z}/p^{\phi} \qquad 0 \qquad \mathbb{Z}/p^{\phi} \qquad \dots . \tag{17}$$

The upper right corner of diagram (14) is the 0 homology term of (17).

This means that the $RO(\mathbb{Z}/p)$-graded coefficients of $i_\sharp H\mathbb{Z}$, as calculated from diagram (14), is obtained from the $RO(\mathbb{Z}/p)$-graded coefficients of $H\mathcal{A}_{\mathbb{Z}/p}$ by replacing, for $k > 0$, Borel cohomology with Borel homology. Thus, we get $I \cong \mathbb{Z}$ for $k > 0$ even and $\ell = -k/2$ and $\mathbb{Z}/p$ for $k > 0$ odd and $\ell > -(k+1)/2$. This is easily confirmed using standard the Segal-tom Dieck splitting.

The only non-trivial step of the Hodge filtration for $G=\mathbb{Z}/p$ is

$$F^1HR_*(X^{\mathbb{Z}/p})\subseteq F^0HR_*(X^{\mathbb{Z}/p})=HR_*(X^{\mathbb{Z}/p}). \tag{18}$$

Given the definition, (18) can be described as the imae of the composition

$$\begin{array}{ccc} (H(\mathcal{A}_{\mathbb{Z}/p}\otimes R)\wedge S^{\infty\beta}\wedge X)_n^{\mathbb{Z}/p} & & \\ \downarrow & & \\ (H\mathcal{A}_{\mathbb{Z}/p}\otimes R)\wedge E\mathbb{Z}/p_+\wedge X)_{n-1}^{\mathbb{Z}/p} & \xleftarrow{\cong} & HR_{n-1}(E\mathbb{Z}/p_+\wedge_{\mathbb{Z}/p}X) \\ & & \downarrow \\ & & HR_{n-1}X \end{array} \tag{19}$$

where the vertical maps come from the isotropy separation exact triangle

$$E\mathbb{Z}/p_+\to S\to S^{\infty\beta}.$$

In the special case $X=S^{k\beta}$, $k\in\mathbb{Z}$, the above calculation implies that we have precisely

$$F^1HR_*((S^{k\beta})^{\mathbb{Z}/p})=HR_{>0}(S^{k\beta}). \tag{20}$$

In particular, for $R=\mathbb{Z}/p$, for $k>0$, this can be restated as

$$H\mathbb{Z}/p_*((S^{-k\beta})^{\mathbb{Z}/p})=H\mathbb{Z}/p_*(C(S\to B\mathbb{Z}/p^{\infty}_{-k}))$$

where on the right hand side, we have the cofiber of the standard map from the 0-sphere tp tje stunted lens space. It is well known that the Steenrod operations cross from positive to negative homological dimensions, so they do not preserve the Hodge filtration.

The case $p=2$ is essentially the same except that $\beta=2\alpha$ where α is the real sign representation. (See also [9].)

2.3. **Primary cyclic groups.** When $G=\mathbb{Z}/(p^m)$ is a primary cyclic group, the situation is essentially analogous. By Quillen's Theorem A, the diagram $\mathcal{D}_G$ can be replaced by its subdiagram where

$$[H:K]\leq p$$

(a "staircase" diagram). It then follows that analogously to (18), the only relevant piece of information in the Hodge filtration is

$$F^1HR_*(X^{\mathbb{Z}/p^m})\subseteq F^0HR_*(X^{\mathbb{Z}/p^m})=HR_*(X^{\mathbb{Z}/p^m}). \tag{21}$$

In turn, the F^1 is the (not generally direct) sum of the images of all its terms with

$$[H:K]=p.$$

Consider $\mathbb{Z}/p^k$ as a factor group of $\mathbb{Z}/p^m$. Then we can characterize these images as the images of the compositions
(22)

$$\begin{array}{ccc}
(H(\mathcal{A}_{\mathbb{Z}/p^k}\otimes R)\wedge \widetilde{E\mathbb{Z}/p^k}\wedge X)_n^{\mathbb{Z}/p} & & \\
\downarrow & & \\
(H\mathcal{A}_{\mathbb{Z}/p^k}\otimes R)\wedge E\mathbb{Z}/p^k_+\wedge X)_{n-1}^{\mathbb{Z}/p} & \xleftarrow{\cong} & HR_{n-1}(E\mathbb{Z}/p^k_+\wedge_{\mathbb{Z}/p^m} X) \\
 & & \downarrow \\
 & & HR_{n-1}X
\end{array}$$

where the vertical maps coe from the istropy separation distinguished triangle

$$E\mathbb{Z}/p^k_+ \to S \to \widetilde{E\mathbb{Z}/p^k}.$$

2.4. **The general case.** Let us now assume that G is any finite group, not necessarily abelian. Formulating our construction in this case would at first appear to encounter a numer of difficulties caused by the presence of non-normal subgroup. It turns out, however, that a closely analogous construction makes sense. if we develop the right concepts.

The first notion we have to discuss are homotopy limits over equivariant diagrams of spectra. (The case of spectra with additional structure, such as E_∞ ring, is completely analogous.) Using the setup of Lewis-May spectra [12], we can consider an equivariant diagram $\mathcal{D}$ of G-$\mathcal{U}$-spectra where $\mathcal{U}$ is a fixed complete universe. This means that the source of $\mathcal{D}$ is a small category on which G acts by a group of automorphisms, and the functor into G-$\mathcal{U}$-spectra is G-equivariant. This means that to an object α of $\mathcal{D}$ of isotropy H, there is assigned an $N(H)$-equivariant spectrum X_α indexed on the universe $\mathcal{U}$, considered as a complete $N(H)$-universe. Morphisms are required to be equivariant with respect to their isotropy groups.

We can define

$$\lim_{\leftarrow}\mathcal{D}$$

for an equivariant diagram of G-$\mathcal{U}$=spectra as described above. This is defined as the right adjoint to the *constant* equivariant diagram, which, on a given source, assigns the same $\mathcal{U}$-G-spectrum X to every object, and Id_X to every morphism.

We can now define the homotopy limit of an equivariant diagram of G-spectra as the total right derived functor of this construction. Explicitly, this can be computed by taking the barycentric subdivision of the source of $\mathcal{D}$, whose set of objects is given by tuples of composable morphisms, and morphisms are given by composition resp. omitting the first or last moprhism. The functor is given by the first object in the tuple.

Now we can "resolve" the barycentric subdivision by sending each composable n-tuple of morphisms to $F(\Delta^n, X_\alpha)$ where α is the source of the first morphism of the tuple and Δ^n is the standard n-simplex.

Now for the case of a non-abelian finite group G, the source of our equivarant diagram consiss of the poset of subgroups $H \subseteq K$ of G where $(H, K) \subseteq (H', K')$ means

$$H' \subseteq H \subseteq K \subseteq K'$$

where G acts by conjugation. The value of the diagram on an object $H \subseteq K$ can be described as

$$H\underline{\pi}_0(\Sigma^\infty \widetilde{E\mathcal{F}[H]}) \wedge \widetilde{E\mathcal{F}[K]}. \tag{23}$$

We note that while we do not technically exclude the case of H being a non-normal subgroup of K, the isotropy of the pair (H, K) is the intersection of normalizers $N(H) \cap N(K)$, which, in the case of H not normal in K, does not contain K, so the term (23), which is an $N(H) \cap N(K)$-equivariant spectrum on the given universe, is contractible, and thus these terms do not contribute to the diagram.

With this caveat, the proof of Theorem 3 proceeds in the same way as in the abelian case and so does the definition of the Hodge filtration. Concrete calculations in the non-abelian case are, of course, much hrader to complete.

3. Some peculiarities of ordinary equivariant homology

In this section, we shall exhibit some manifestations of "non-commutativity" of the pushforward $i_\sharp H\mathbb{Z}$. To this end, we shall specialize to the case $H = \mathbb{Z}/p$, and we shall take a closer look at the chain complex of Mackey functors (16).

3.1. **Non-commutativity of chains.** Since (16) is the chain complex of $S^{\infty\beta}$ with coefficients in the Burnside Mackey functor $\mathcal{A}$, it automatically has a structure of an E_∞-algebra in the category of chain complexes of Mackey functors. We will see that it does not have a strictly graded-commutative differential graded algebra model. This is, as usual, done by showing that there are non-trivial Dyer-Lashof operations. In fact, Dyer-Lashof operations apply to E_∞-$\mathbb{F}_p$-algebras, so we take $\mathbb{Z}/p$-fixed points and reduce modulo p, both of which preserve E_∞-algebra as well as graded-commutative DGA structures. After applying these steps, we are left with the E_∞-algebra in $\mathbb{F}_p$-modules given as

$$(\widetilde{C}_*(S^{\infty\beta};\mathbb{F}_p))^{\mathbb{Z}/p} \tag{24}$$

To show that the E_∞-algebra (24) has non-trivial Dyer-Lashof operations, we note that (24) has a periodicity class t in second homology; if we invert this class, we obtain simply the Tate cohomology chain complex (the chain realization of ordinary Tate cohomology) of $G = \mathbb{Z}/p$ with coefficients in $\mathbb{F}_p$.

The Dyer-Lashof operations of the Tate chain complex, in turn, come from those in the Borel cohomology chain complex, which is simply

$$C^*(B\mathbb{Z}/p;\mathbb{F}_p),$$

in which the Dyer-Lashof operations are sunply the Steenrod operations of $B\mathbb{Z}/p$, and therefore are non-trivial. By the Whitney formula, then, the Dyer-Lashof operations are also non-trivial in the homology of (24). For example, the total Dyer-Lashof operation of t is

$$t^p(1+t^{p-1})^{-1}.$$

Thus, in particular, (24) does not have a strictly graded-commutative DGA model.

3.2. **The forgetful functor from Mackey functor to G-spectra is not faithful.** The reason why diagram (14) exists on the level of $\mathbb{Z}/p$-spectra but cannot be fully realized on chain level is the existence of two chain models of the lower right corner (before or after applying the functor $\phi_{\mathbb{Z}/p}$. These two models are related by a "tilting" which works spectrally, but not on chain level.

To demonstrate this, we have already seen that the 0-stage of the Postnikov tower of the $\mathbb{Z}/p$-spectral realization of (16) splits off (in fact, we also saw that by arguing more carefully, the splitting can be chosen to be a morphism of E_∞-ring spectra). This is, again, simply because the splitting exists in $S^{\infty\beta}$-modules, the derived category of which is equivalent to the category of non-equivariant spectra by taking $\mathbb{Z}/p$-fixed points: in non-euivariant spectra, we are dealing simply with a generalized Eilenberg-Mac Lane spectrum, which therefore splits as a wedge sum of the Eilenberg-Mac Lane spectra corresponding to is homotopy groups.

We will now show, however, that the first k-invariant of the chain complex of Mackey functors (16), which, by (17), lies in

$$Ext^3_{\mathcal{A}}(\mathbb{Z}^{\phi}, \mathbb{Z}/p^{\phi}), \tag{25}$$

is non-zero. To do this calculation, we recall the constant and co-constant $\mathbb{Z}/p$-Mackey functors $\underline{\mathbb{Z}}$, $\overline{\mathbb{Z}}$, and also the $\mathbb{Z}/p$-Mackey functor $\widetilde{\mathcal{L}}_{p-1}$, which is the integral reduced regular $\mathbb{Z}/p$-representation $\mathcal{L}_{p-1}$ in isotropy $\{0\}$ and 0 in isotropy $\mathbb{Z}/p$. Then one has a 4-periodicity given by short exact sequences of $\mathbb{Z}/p$-Mackey functors

$$0 \to \mathbb{Z}^{\phi} \to \mathcal{A} \to \underline{\mathbb{Z}} \to 0,$$

$$0 \to \overline{\mathbb{Z}} \to \mathcal{A} \to \mathbb{Z}^{\phi} \to 0,$$

$$0 \to \widetilde{\mathcal{L}}_{p-1} \to \underline{\mathcal{L}}_p \to \overline{\mathbb{Z}} \to 0,$$

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathcal{L}}_p \to \widetilde{\mathcal{L}}_{p-1} \to 0.$$

The $\tau_{\leq 2}$ (graded homologically) of (16) can be represented by the chain complex

$$\mathcal{A} \xleftarrow{1} \underline{\mathcal{L}}_p \xleftarrow{1-\gamma} \underline{\mathcal{L}}_p \longleftarrow \overline{\mathbb{Z}} \tag{26}$$

whose homology is given by the first three entries of (17). The $\tau_{\leq 0}$ is $\mathbb{Z}^{\phi}$ which is equivalent to the chain complex

$$\mathcal{A} \xleftarrow{1} \underline{\mathcal{L}}_p \xleftarrow{1-\gamma} \underline{\mathcal{L}}_p \longleftarrow \underline{\mathbb{Z}}. \tag{27}$$

A chain map from (26) to (27) is identity on the first three terms together with the inclusion $\kappa : \overline{\mathbb{Z}} \to \underline{\mathbb{Z}}$ which is p in isotropy $\mathbb{Z}/p$ and identity in isotropy $\{0\}$. We conclude that our k-invariant in (25) is equal, by the above periodicity, to the element of

$$Hom_{\mathcal{A}}(\underline{\mathbb{Z}}, \mathbb{Z}/p^{\phi})$$

given by the cokernel of κ, which is non-zero.

Thus, the first k-invariant of (16) is non-zero in the derived category of Mackey functors (where it lies in (25)) but vanishes in the derived category of $\mathbb{Z}/p$-equivariant spectra. We conclude therefore that the forgetful functor from the derived category of $\mathbb{Z}/p$-Mackey functors to the derived category of $\mathbb{Z}/p$-equivariant spectra is not faithful.

3.3. **Non-existence of spectral Mackey functors.** In this subsection, for simplicity, let us specialize to $G = \mathbb{Z}/2$. In this case, the isomorphism $A_{\mathbb{Z}/2} \cong \mathbb{Z}[\mathbb{Z}/2]$ sending the free orbit to $1+\gamma$ (where γ is the generator) gives a curious ivolutiony on $\mathbb{Z}/2$-Mackey functors swapping the free with the fixed orbit and restriction with corestriction. This involution swaps $\mathcal{A}_{\mathbb{Z}/2}$ with the free Mackey functor on one $\mathbb{Z}/2$-free generator $\underline{\mathcal{L}}_2$, and Mackey functors which are 0 on the free orbit with those which are 0 on the fixed orbit.

Now we can observe that the derived category of spectrra which are contractible on the free orbit is equivalent, via taking fixed points, to the derived category of non-equivariant spectra. We claim, on the other hand, the following

1. **Proposition.** *A $\mathbb{Z}/2$-equivariant 2-completion of a bounded below spectrum X of finite type with contractible $\mathbb{Z}/2$-fixed points is a $\mathbb{Z}/2$-equivariant generalized Eilenberg-Mac Lane spectrum.*

Proof. Denote by M any abelian group and by M_f the $\mathbb{Z}/2$-Mackey functor which is 0 on the fixed orbit and M (on which γ acts by minus) on the free orbit. one has (cf. [9])

$$HM_f = \Sigma^{1-\alpha} H\underline{M}$$

where α is the real sign representation. Thus, if X is as described in the statement of the Proposition, then the terms of the Postnikov tower of $\Sigma^{\alpha-1}X$ are of the form $H\underline{M}$ where M is a finitely generated $\mathbb{Z}_2$-module. However, the only non-trivial morphisms between the $\mathbb{Z}$-graded suspensions of such spectra are Bocksteins ([9, 3]). $\square$

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